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Generalizing Stretch Lines for Surfaces with Boundary

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In 1986, William Thurston introduced the celebrated (asymmetric) Lipschitz distance on the Teichmüller space of closed or punctured surfaces. We extend his theory to the Teichmüller space of surfaces with boundary endowed with the arc distance. We construct a large family of geodesics for the Teichmüller space of a surface with boundary, generalizing Thurston’s stretch lines. We prove that the Teichmüller space of a surface with boundary is a geodesic and Finsler metric space with respect to the arc distance. As a corollary, we find a new class of geodesics in the Teichmüller space of a closed surface that are not stretch lines in the sense of Thurston.

1 Introduction

In this paper we will study the geometry of the Teichmüller space of an oriented surface of finite type with non-empty boundary when it is endowed with the *arc distance*. This is an asymmetric distance, which generalizes the celebrated *Thurston’s asymmetric distance* on the Teichmüller space of a closed surface defined by William P. Thurston.

1.1 Thurston’s theory for closed or punctured surfaces

In [20], Thurston defines two asymmetric distances on the Teichmüller space of a closed or punctured surface, which naturally mimics the Teichmüller distance in a hyperbolic

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setting: the distance d_{Th}, which encodes the changes in the length spectrum of simple closed curves; and the Lipschitz distance d_{Lh}, which measures the optimal Lipschitz constant of a homeomorphism isotopic to the identity (for precise definitions, see Section 2.4). Following the analogy with the Teichmüller distance, Thurston constructs a family of paths, called *stretch lines*, which are geodesics for both distances. Using these paths, Thurston proves that the two distances always coincide and turn the Teichmüller space into a geodesic Finsler metric space (see Section 2.5). Understanding the structure and behavior of *all* the geodesics for Thurston's distance is still an open problem. The geometry of Thurston's distance of the Teichmüller space of closed or punctured surfaces was further studied by many authors, including Bonahon [2, 3], Papadopoulos [12], Théret [19], Walsh [22], Dumas–Lenzhen–Rafi–Tao [6], Lenzhen–Rafi–Tao [9, 10], and Choi–Rafi [4].

1.2 The theory for surfaces with boundary

In this paper we study similar asymmetric distances on the Teichmüller space Teich(S) of a surface S with non-empty boundary. The case of surfaces with boundary is particularly interesting, see, for example, the beautiful applications by Guéritaud–Kassel [7] on proper affine actions of free groups and Margulis spacetime (notice that the presence of a non-empty boundary is crucial in their work). In the case of surfaces with boundary, Thurston's original formula d_{Th} does not give a distance anymore (see Parlier [17], Papadopoulos–Théret [14], and Section 2.6). In spite of this, Liu–Papadopoulos–Su–Théret [11] defined a new distance, the *arc distance* d_A, which considers the length spectrum of simple closed curves and simple proper arcs orthogonal to the boundary (for details, see Section 2.4). A few examples of geodesics for d_A were given by Papadopoulos–Théret [15] and Papadopoulos–Yamada [16]. Alessandrini–Liu–Papadopoulos–Su [1] studied the close relationship between the arc distance and Thurston's compactification.

The arc distance is the main object of study in this paper. Motivated by [1, 15, 16], we will study the property of the metric space (Teich(S), d_A). We will prove that it is a geodesic Finsler metric space. We will construct some special paths, the *generalized stretch lines*, which mimic the properties of Thurston's original stretch lines in this new setting. Our results will be useful in work in progress by Calderon–Farre to produce their shear-shape coordinates for Teichmüller space. In their recent preprint [8], Huang–Papadopoulos study similar questions in the special case of the one-holed torus in the different setting of Teichmüller spaces with fixed boundary length. All the results in this paper were announced at the Oberwölfach conference “New Trends in Teichmüller Theory and Mapping Class Groups” in 2018 (see [5]).

1.3 Our results

Let S be an orientable surface of finite type with non-empty boundary. In this paper we introduce a new asymmetric distance on Teich(S), denoted by $d_{L\partial}$, which measures the optimal Lipschitz constant of a continuous map preserving ∂S isotopic to the identity. We now have three distances on Teich(S), which satisfy the following inequalities (for precise definitions and statements, see Section 2.4):

$$d_A \leq d_{L\partial} \leq d_{Lh}\,.$$

In analogy with Thurston’s theory, we will construct a large family of geodesics for the two distances, d_A and $d_{L\partial}$, which we call generalized stretch lines. For any two points on the same generalized stretch line we will construct an optimal Lipschitz map, which we call a generalized stretch map. The constructions of generalized stretch lines and stretch maps are the most important contributions of this paper. This construction can be summarized in the following statement (the terminology will be introduced later in Sections 2, 3, and 5).

Theorem 1.1. Let S be a surface with non-empty boundary and fix $X \in$ Teich(S). For every maximal lamination λ on X and for every $t \geq 0$ there exists $X_\lambda^t \in$ Teich(S) and a Lipschitz map $\Phi^t : X \to X_\lambda^t$, called *generalized stretch map*, with the following properties:

1. $X_\lambda^0 = X$;
2. $\mathrm{Lip}\,(\Phi^t) = e^t$;
3. $\Phi^t(\partial X) = \partial X_\lambda^t$;
4. Φ^t stretches the arc length of the leaves of λ by the factor e^t;
5. for every geometric piece $\mathcal{G}$ in $X \setminus \lambda$, the map Φ^t restricts to a generalized stretch map $\phi^t : \mathcal{G} \to \mathcal{G}_t$ as described in Lemmas 5.5, 5.8;
6. if λ contains a non-empty measurable sublamination, we have

$$\mathrm{Lip}(\Phi^t) = \min\,\{\mathrm{Lip}(\psi) \mid \psi \in \mathrm{Lip}_0(X, X_\lambda^t), \psi(\partial X) \subset \partial X_\lambda^t\},$$

where $\mathrm{Lip}_0(X, Y)$ is the set of all Lipschitz maps homotopic to the identity.

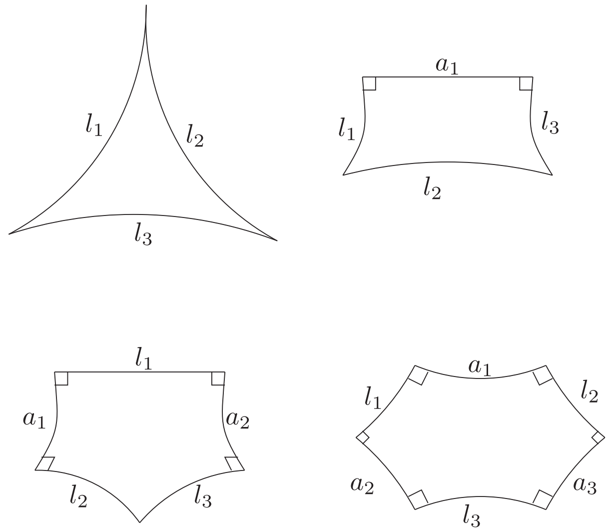


Fig. 1. The four geometric pieces: the edges a_i correspond to segments in ∂X and the edges l_i correspond to leaves of λ.

Corollary 1.2. For every $X \in \text{Teich}(S)$ and every maximal lamination λ on X, if λ contains a non-empty measurable sublamination then the *generalized stretch line*

$$s_{X,\lambda} : \mathbb{R}_{\geq 0} \longrightarrow \text{Teich}(S)$$
$$t \mapsto X_\lambda^t$$

is a geodesic path parametrized by arc-length for both d_A and $d_{L\partial}$.

Our construction presents new challenges when compared with Thurston's work. In Thurston's case of closed or punctured surfaces, every maximal lamination decomposes the surface into ideal triangles. Thurston constructs the stretch map between two ideal triangles explicitly via the horocyclic foliation. In the case of surfaces with boundary a maximal lamination decomposes the surface in geometric pieces of four different types (see Fig. 1). Unlike Thurston [20], we do not construct explicit maps between the geometric pieces. Instead, we use a trick of its own interest, which allows to "average" two Lipschitz maps. Our *average map* will be a Lipschitz map whose Lipschitz constant is bounded above by the average of the two Lipschitz constants. Our construction is obtained by adapting a result of Guéritaud–Kassel [7]. Using generalized stretch lines, we prove the following:

Theorem 1.3. The space $(\mathrm{Teich}(S), d_A)$ is a geodesic metric space. Every two points $X, Y \in \mathrm{Teich}(S)$ can be joined by a segment that is geodesic for both d_A and $d_{L\partial}$ and is a finite concatenation of generalized stretch segments.

Corollary 1.4. The arc distance d_A is induced by a Finsler metric on $\mathrm{Teich}(S)$.

We find that d_A and $d_{L\partial}$ coincide, this completes our generalization of Thurston's [20, Theorem 8.5].

Corollary 1.5. Given $X, Y \in \mathrm{Teich}(S)$, there exists a continuous map $\phi \in \mathrm{Lip}_0(X, Y)$, with $\phi(\partial X) \subset \partial Y$ and with optimal Lipschitz constant such that $\log(\mathrm{Lip}(\phi)) = d_A(X, Y)$. In particular, we have

$$d_A(X, Y) = d_{L\partial}(X, Y).$$

As a byproduct of our constructions, we also find new geodesics for the Teichmüller space of closed or punctured surfaces endowed with Thurston's distance. Indeed, Liu–Papadopoulos–Su–Théret [11] proved that the doubling map

$$\jmath : (\mathrm{Teich}(S), d_A) \ni X \hookrightarrow X^d \in (\mathrm{Teich}(S^d), d_{Th})$$

is an isometry. By doubling our generalized stretch lines, we can construct many new geodesics for $(\mathrm{Teich}(S^d), d_{Th})$ that lie completely in the submanifold of *symmetric hyperbolic structures*.

Corollary 1.6. The map $(\mathrm{Teich}(S), d_A) \hookrightarrow (\mathrm{Teich}(S^d), d_{Th})$ is a geodesic embedding.

Notice that Thurston's construction of stretch lines in general breaks the symmetry of hyperbolic structures, see for instance the examples by Théret [19]. Our construction, instead, provides new geodesics that preserve symmetric hyperbolic structures.

Corollary 1.7. Let $X \in \mathrm{Teich}(S)$ and let λ be a maximal lamination of X containing a measurable sublamination with at least one leaf orthogonal to the boundary of X. Then, the line $t \mapsto (X^t_\lambda)^d \in \mathrm{Teich}(S^d)$ is a geodesic for $(\mathrm{Teich}(S^d), d_{Th})$ that is not a stretch line in the sense of Thurston [20].

1.4 Sketch of the proof of Theorem 1.1

Let $X \in \mathrm{Teich}(S)$ and λ be a maximal lamination on X. We want to construct a generalized stretch line starting from X and directed by λ, that is, for every $t \geq 0$ we want to construct $X^t_\lambda \in \mathrm{Teich}(S)$ satisfying the properties of Theorem 1.1.

1.4.1 *Geometric pieces*

As a 1st step we characterize the connected components of $X \setminus \lambda$, that is, the geometric pieces. We will see in Proposition 3.2 that there are only four types of pieces (see Fig. 1).

Given a geometric piece $\mathring{\mathcal{G}} \subset X \setminus \lambda$ we define a suitable generalized stretch map $\phi^t : \mathcal{G} \to \mathcal{G}^t$ from the original piece $\mathcal{G}$ to its "stretched" analogue $\mathcal{G}^t$. The map has optimal Lipschitz constant $\mathrm{Lip}(\phi^t) = e^t$. When $\mathcal{G}$ is an ideal triangle, we use the homeomorphism defined by Thurston. In the other cases we use an implicit construction, which generalizes an argument by Guéritaud–Kassel [7] (see Sections 4 and 5).

1.4.2 *Decomposition of X*

If λ has no leaves that hit the boundary of X, all the geometric pieces are triangles and Theorem 1.1 follows by Thurston [20]. We are interested in the case where at least one leaf of λ is orthogonal to ∂X. We define the boundary block of λ in X in Section 6.1 as the (possibly disconnected) subsurface given by the union of all the non-triangular geometric pieces (Fig. 14):

$$B = \bigsqcup \{\mathcal{G}_i \mid \mathcal{G}_i \text{ is a geometric piece that is not an ideal triangle } \} \subseteq X\,.$$

The boundary block is a complete hyperbolic surface of finite volume, whose boundary might be non-compact. It is equipped with a finite maximal lamination $\lambda_B \subset \lambda$. The boundary ∂B will contain a finite union of *cycles* c_j, each determining a *crown* C_j as in Fig. 13 (Definitions 6.1 and 6.2, respectively). Denote the union of all such crowns by C and the complement of C in B by $B_C := \overline{B \setminus C} \subset B$. Denote by $X_C := \overline{X \setminus B_C} \subset X$ the complement of B_C in X, defined in Section 6.1, see Fig. 14. The surface X_C is equipped with a lamination $\lambda_{X_C} \subset \lambda$ where no leaf is orthogonal to ∂X_C. We have the following commutative diagram, where $\lambda = \lambda_B \cup \lambda_{X_C}$:

$$\begin{tikzcd}
 & (B, \lambda_B) \arrow[dr] & \\
C \arrow[ur] \arrow[dr] & & (X, \lambda) \\
 & (X_C, \lambda_{X_C}) \arrow[ur] &
\end{tikzcd}$$

Every arrow is the canonical inclusion, which is also a Riemannian isometry.

1.4.3 *Strategy to define X^t_λ*

For $t \geq 0$ we construct suitable complete hyperbolic surfaces B^t, C^t, and $(X_C)^t$ homeomorphic to B, C, and X_C, respectively (see Sections 6.2, 7.4, and 8). The new surfaces come with preferred Riemannian isometries $\iota^t : C^t \hookrightarrow B^t$ and $h^t : C^t \hookrightarrow X^t_C$. We then define X^t_λ as follows:

$$X^t_\lambda := B^t \bigsqcup (X_C)^t / \sim ,$$

where $\iota^t(z) \sim h^t(z)$ for every $z \in C^t$. The quotient projection $\pi : B^t \bigsqcup (X_C)^t \to X^t_\lambda$ restricts to Riemannian isometries on $(X_C)^t$ and B^t (Proposition 8.1):

$$\begin{tikzcd}
 & B^t \arrow[dr, "\pi_|"] & \\
C^t \arrow[ur, "\iota^t"] \arrow[dr, "h^t"'] & & X^t_\lambda \\
 & (X_C)^t \arrow[ur, "\pi_|"'] &
\end{tikzcd}$$

The generalized stretch map $\Phi^t : X \to X^t_\lambda$ is defined by glueing together suitable stretch maps $\beta^t : B \to B^t$ and ψ^t from an open dense subset of X_C to $(X_C)^t$, with the required properties. The details about the construction of (B^t, β^t) are given in Section 6.2. The details about the construction of $((X_C)^t, \psi^t)$ are given in Sections 8.1 and 7.4. The details about how to glue the maps β^t and ψ^t are discussed in Section 8.2.

1.5 Open problems

Our work leads us to conjecture that the three natural distances $d_A, d_{L\partial}, d_{Lh}$ on Teich(S) are all equal:

Conjecture 1.8. For every $X, Y \in$ Teich(S) we have

$$d_A(X, Y) = d_{Lh}(X, Y) \ .$$

Notice that our generalized stretch map Φ^t is a homeomorphism if and only if its restriction to each geometric piece $\Phi^t|_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}^t$ is also a homeomorphism. Since our construction of the maps is not explicit, we cannot tell whether they are injective.

1.6 Organization of the paper

This paper is organized as follows. In Section 2, we introduce the main definitions that we use throughout the paper and we give a more detailed account of the theory of asymmetric distances on Techmüller spaces. In Section 3, we introduce the notion of geodesic laminations, maximal, and measured laminations for surfaces with boundary. In Section 4 we describe an averaging procedure for Lipschitz maps between convex hyperbolic surfaces. In Section 5, we use it to construct optimal Lipschitz maps between the geometric pieces. After these preliminary sections, we describe the construction of our generalized stretch lines. In Sections 6 and 7, we construct some auxiliary surfaces (the boundary block and the triangulated surface) and we describe how to stretch them. In Section 8, we glue the stretched boundary block and part of the stretched triangulated surface together in a suitable way, in order to construct the generalized stretch lines. This will prove Theorem 1.1. In Section 9, we prove all the other results stated in the introduction.

2 Background

In this section, we introduce the main definitions that we use throughout the paper and we give an account of the theory of asymmetric distances on Techmüller spaces.

2.1 Hyperbolic surfaces

We start by recalling some basic definitions about hyperbolic surfaces. We denote the *hyperbolic plane* by

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\},$$

endowed with the Riemannian metric

$$g_{\mathbb{H}^2} = \frac{dx^2 + dy^2}{y^2}, \text{ with } z = x + iy.$$

The *hyperbolic half-plane* is the subset

$$\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) > 0\},$$

where the positive y-axis is its geodesic boundary.

Definition 2.1 (Hyperbolic surface). A *hyperbolic surface* is a Riemannian manifold X (possibly with boundary) where every point has a neighborhood isometric to an open subset of the hyperbolic half-plane. A *complete hyperbolic surface* is a hyperbolic surface that is complete as a metric space.

Definition 2.2 (Convex hyperbolic surface). A *convex hyperbolic surface* is a connected hyperbolic surface whose universal covering is isometric to a convex subset of the hyperbolic plane.

Note that the boundary of a complete hyperbolic surface is a union of geodesics (circles or infinite geodesic lines). Moreover, the connected components of a complete hyperbolic surface are always convex. Examples of complete hyperbolic surfaces are ideal polygons in the hyperbolic plane.

Definition 2.3 (Finite hyperbolic surface). A *finite hyperbolic surface* is a complete hyperbolic surface with finite volume and compact boundary.

For such finite hyperbolic surfaces, every boundary component is a closed geodesic, topologically a circle, and every puncture is isometric to a cusp.

Definition 2.4 (Local isometry). A *local isometry* between two hyperbolic surfaces X and Y is a local diffeomorphism such that the pull-back of the metric on Y is equal to the metric on X.

Remark 2.5. In this paper, we will need to consider many local isometries that are 1-1 but still are not isometric embeddings in the sense of metric spaces, that is, they do not preserve the distances between pairs of points. This is because their image is not a convex subset of the target surface.

2.2 Teichmüller space

In this paper we will denote by S an orientable surface of finite type of genus g, with $b \geq 0$ compact boundary components and $p \geq 0$ punctures. The boundary of S will be denoted by ∂S. We assume the Euler characteristic $\chi(S) = 2 - 2g - b - p$ to be negative.

Definition 2.6 (Hyperbolic structure). A *hyperbolic structure* on S is a pair (X, m), where X is a finite hyperbolic surface and $m : S \to X$ is a homeomorphism, called the *marking*.

Definition 2.7 (Teichmüller space). The *Teichmüller space* Teich(S) is the space of all hyperbolic structures on S up to isometries that commute with the markings (up to homotopy). We will denote an element $[(X, m)] \in \text{Teich}(S)$ by X for short.

The Teichmüller space Teich(S) is diffeomorphic to $\mathbb{R}^{6g-6+2p+3b}$. Let S^d be the surface obtained doubling S along its boundary, and let $\sigma : S^d \to S^d$ be the associated involution. A hyperbolic structure on S can be equivalently defined as a hyperbolic structure on S^d whose isometry group contains σ.

Definition 2.8 (Doubling embedding). If X is a hyperbolic structure on S, its *double* X^d is the hyperbolic structure on S^d obtained by doubling X. This gives an embedding

$$\text{Teich}(S) \ni X \hookrightarrow X^d \in \text{Teich}(S^d).$$

2.3 Curves and arcs

A simple closed curve in S is *trivial* if is either null-homotopic or homotopic to a puncture of S. We will denote by $\mathcal{C}$ the set of homotopy classes of non-trivial simple closed curves on S, and by $\mathcal{B}$ the boundary components of ∂S. We recall that for every

$X \in \mathrm{Teich}(S)$ and for every $\gamma \in \mathcal{C}$, there is a unique X-geodesic curve in the homotopy class γ, which is the shortest curve in γ. We will define the *length* $\ell_X(\gamma)$ to be the length of this geodesic curve.

A *proper arc* in S is a continuous map $\alpha : [0,1] \to S$ with $\{\alpha(0), \alpha(1)\} \subset \partial S$. Our arcs are *unoriented*, that is, we will consider $\alpha(t)$ equivalent to $\alpha(1-t)$ A proper arc is *simple* if it is injective. Two proper arcs are *properly homotopic* if they are connected by a homotopy where the extremes of the arcs never leave ∂S at any time.

Definition 2.9 (Essential arc). An *essential arc* is a proper arc that is not properly homotopic to a proper arc contained in ∂S. We will denote by $\mathcal{A}$ the set of proper homotopy classes of essential simple arcs. Recall that the elements of $\mathcal{A}$ are unoriented.

It is well known that for every $X \in \mathrm{Teich}(S)$ and for every $\alpha \in \mathcal{A}$, there is a unique X-geodesic arc in the proper homotopy class α that is orthogonal to ∂S. This arc is the shortest in its proper homotopy class; its length is denoted by $\ell_X(\alpha)$.

2.4 Five functionals

The length functions $\ell_X(\cdot)$ of curves and arcs can be used to compare two hyperbolic structures and, in some cases, define distances on Teichmüller spaces. We will be interested in the following functionals:

$$d_{Th}(X,Y) = \sup_{\gamma \in \mathcal{B} \cup \mathcal{C}} \log \frac{\ell_Y(\gamma)}{\ell_X(\gamma)}, \tag{1}$$

$$d_A(X,Y) = \sup_{\delta \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}} \log \frac{\ell_Y(\delta)}{\ell_X(\delta)}. \tag{2}$$

Another natural way to compare two elements $X, Y \in \mathrm{Teich}(S)$ is to consider Lipschitz maps between them. Let $\mathrm{Lip}_0(X,Y)$ be the set of Lipschitz maps between X and Y that commute with the markings up to homotopy. Denote by $\mathrm{Lip}(\phi)$ the Lipschitz constant of a map ϕ. We will consider the following functionals:

$$d_L(X,Y) = \inf\{\log \mathrm{Lip}(\phi) \mid \phi \in \mathrm{Lip}_0(X,Y)\}; \tag{3}$$

$$d_{L\partial}(X,Y) = \inf\{\log \mathrm{Lip}(\phi) \mid \phi \in \mathrm{Lip}_0(X,Y), \phi(\partial X) \subset \partial Y\}; \tag{4}$$

$$d_{Lh}(X,Y) = \inf\{\log \mathrm{Lip}(\phi) \mid \phi \in \mathrm{Lip}_0(X,Y), \phi \text{ is a homeomorphism}\}. \tag{5}$$

It is immediate that the above five functionals satisfy the following inequalities:

$$\begin{array}{ccccc} d_{Th}(X,Y) & \leq & d_L(X,Y) & \leq & d_{Lh}(X,Y) \\ \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \| \\ d_A(X,Y) & \leq & d_{L\partial}(X,Y) & \leq & d_{Lh}(X,Y)\,. \end{array}$$

It is not difficult to see that they all satisfy the triangular inequality

$$d_*(X,Z) \leq d_*(X,Y) + d_*(Y,Z)\ ,$$

and they are not symmetric

$$\exists X,Y : d_\star(X,Y) \neq d_\star(Y,X)\,.$$

A method to produce such X,Y is given in [20, Section 2] and works also for surfaces with boundary. In the following we will discuss when the axiom of positivity

$$d_\star(X,Y) \geq 0 \text{ and } d_\star(X,Y) = 0 \Leftrightarrow X = Y$$

holds, that is, the functionals actually define asymmetric distances. We will see that the answer depends on whether ∂S is empty or not.

2.5 Closed or punctured surfaces

The case of closed or punctured surfaces was the case originally studied by Thurston [20]. He introduced the functionals d_{Th} and d_{Lh} and proved that they satisfy the positivity axiom.

One of the main results of [20] is that given $X,Y \in \mathrm{Teich}(S)$, there exists a homeomorphism $\phi \in \mathrm{Lip}_0(X,Y)$ such that $\log(\mathrm{Lip}(\phi)) = d_{Th}(X,Y)$. This implies

$$d_{Th} = d_L = d_{Lh}\ .$$

This distance is usually called *Thurston's asymmetric distance*, or the *Lipschitz distance*. A crucial step in the proof is the construction of a special family of lines in $\mathrm{Teich}(S)$, the *stretch lines*, which are geodesics for the three distances. Given two points on the same stretch line, there is an optimal Lipschitz homeomorphism between them, the so-called *stretch map*. Using these techniques, given two points $X,Y \in \mathrm{Teich}(S)$ he constructed a geodesic segment between X and Y by concatenating a finite number of such stretch lines. This proves that the Teichmüller space with Thurston's asymmetric distance is a geodesic metric space. Thurston also studied the infinitesimal behaviour of his distance. He proved that it agrees with the asymmetric distance associated with a certain Finsler metric on the Teichmüller space.

2.6 Surfaces with boundary

In the case of surfaces with non-empty boundary, Thurston's functional d_{Th} does not satisfy the axiom of positivity. Indeed, Parlier [17] found two points $X \neq Y \in \mathrm{Teich}(S)$ with $d_{Th}(X, Y) \leq 0$. Papadopoulos–Théret [14] found elements $X \neq Y \in \mathrm{Teich}(S)$ with $d_{Th}(X, Y) < 0$.

The properties of the functional d_{Th} for surfaces with boundary were studied in detail by Guéritaud–Kassel [7], who also introduced the functional d_L. They proved that the two functionals are related as follows:

- if $d_{Th}(X, Y) \geq 0$, then $d_{Th}(X, Y) = d_L(X, Y)$;
- if $d_{Th}(X, Y) < 0$, then $d_L(X, Y) < 0$.

They give applications to the theory of affine actions on $\mathbb{R}^3$ and Margulis space times.

The functional d_A was introduced by Liu–Papadopoulos–Théret–Su [11]. They proved that d_A satisfies the axiom of positivity, therefore it defines an asymmetric distance on $\mathrm{Teich}(S)$, which they called the *arc distance*. They proved the following:

Proposition 2.10 (Liu–Papadopoulos–ThéretSu [11, Corollary 2.8]). The doubling map $(\mathrm{Teich}(S), d_A) \hookrightarrow (\mathrm{Teich}(S^d), d_{Th})$ is isometric.

This proposition will be useful for the present work: whenever possible, we use doubling arguments to reduce our questions to well understood questions about surfaces without boundary. In any case, we remark that it is not possible to construct many geodesics for the arc distance using just doubling arguments. Other properties of the distance d_A were studied in [1, 13, 16].

The functional $d_{L\partial}$ is introduced in this paper in order to interpolate between d_A, d_{Lh}:

$$d_A(X, Y) \leq d_{L\partial}(X, Y) \leq d_{Lh}(X, Y)\,.$$

Here we prove that $d_A = d_{L\partial}$ (Corollary 1.5), that the Teichmüller space with this distance is a geodesic metric space (Theorem 1.3) and that this distance is induced by a Finsler metric (Corollary 1.4).

3 Geodesic Laminations for Surfaces with Boundary

In this section, we will review the main definitions and results about geodesic laminations for surfaces with boundary.

3.1 Maximal laminations

Let $X \in \mathrm{Teich}(S)$. A *geodesic lamination* on X is a (possibly empty) closed subset $\lambda \subset X$ that is a union of pairwise disjoint simple X-geodesics, called the *leaves* of λ, satisfying the following additional condition: if a leaf of λ intersects the boundary, then the leaf must be a boundary component, or it intersects the boundary orthogonally.

It is well known that geodesic laminations do not depend on the hyperbolic structure X, but only on the topological surface S. More precisely, given $X, Y \in \mathrm{Teich}(S)$, for every geodesic lamination λ on X, there exists a unique geodesic lamination λ' on Y and a homeomorphism $f : X \to Y$ consistent with the markings that maps λ to λ'. In light of this, we will often consider geodesic laminations as topological objects on S, without specifying the underlying hyperbolic structure.

A *sublamination* λ' of a geodesic lamination λ is a closed subset $\lambda' \subset \lambda$ that is itself a geodesic lamination. A *maximal lamination* is a geodesic lamination that is maximal with respect to inclusion, that is, it is not a sublamination of a strictly larger geodesic lamination. When $\partial X = \emptyset$, a maximal lamination decomposes X into finitely many ideal triangles. When $\partial X \neq \varnothing$, a maximal lamination also decomposes X into finitely many pieces, in general not always triangles. We will classify them in Proposition 3.2.

Let λ be a geodesic lamination on X. The *double* of λ is the lamination λ^d on X^d obtained by doubling λ. Note that λ^d is maximal if and only if λ is maximal and does not contain leaves orthogonal to ∂X.

Definition 3.1 (Geometric piece). A *geometric piece* is a polygon in $\mathbb{H}^2$ as in Fig. 1:

- an ideal triangle, called *triangular piece*;
- a right-angled quadrilateral with two consecutive ideal vertices, called *quadrilateral piece*;
- a right-angled pentagon with one ideal vertex, called *pentagonal piece*;
- a right-angled hexagon, called *hexagonal piece*.

Proposition 3.2. If λ is a maximal lamination on X, then $X \setminus \lambda$ has $2|\chi(S)| = 4g - 4 + 2p + 2b$ connected components. Each connected component is locally isometric to the interior of a geometric piece, where the edges labeled a_i correspond to segments in ∂X and edges labeled l_i correspond to leaves of λ (see Fig. 1).

Proof. Let C be a connected component of $X\backslash\lambda$. Note that C contains no essential simple closed curve; otherwise, we could extend λ further by adding such a curve. Hence, there are three possibilities for the topology of C: a pair of pants, a cylinder, or a disk. If C were a cylinder or a pair of pants, it would have 2 or 3 ends, but this is impossible: every end of C contains a spike, or a segment of ∂X, or a simple closed curve in X coming from a leaf of λ or an entire boundary component in ∂X. If there is more than one end, we could add one more leaf to λ joining two of them.

We conclude that C is topologically a disk, that is, it is isometric to a hyperbolic polygon whose boundary contains segments in ∂X or leaves of λ and whose vertices are right angled or ideal. Denote by s the number of its ideal vertices and by n the number of segments in ∂X. Given two spikes, a spike and a segment, or two segments, we can join them with a geodesic perpendicular to ∂X that, by maximality, must lie in the boundary of C. Hence, $s+n \leq 3$, and it must be $s+n=3$. Now we can see that the possibilities are $s=3, n=0$ (ideal triangle), $s=2, n=1$ (quadrilateral), $s=1, n=2$ (pentagon), $s=0$, and $n=3$ (right-angled hexagon) as in Fig. 1.

Now we count the number of connected components in $X \setminus \lambda$. Since each one has area π and the surface has area $2\pi|\chi(S)|$, we find $2|\chi(S)|$ connected components. ∎

Proposition 3.3. Every lamination λ can be extended to a maximal lamination by adding finitely many leaves.

Proof. If λ is not maximal then $X \setminus \lambda$ is a finite union of finite-area connected subsurfaces with boundary. Up to extending λ with finitely many simple closed curves, we can assume that each connected component is either a disk, a cylinder or a pair of pants. Since the area of each piece is π, each of its boundary component is a finite polygonal with ideal or right-angled vertices. It can thus be further subdivided with at most finitely many simple essential arcs. ∎

3.2 Transverse measures

Let λ be a geodesic lamination on X and k be an arc transverse to λ. A *transverse isotopy* of k is an isotopy that preserves the transversality of k and such that the endpoints of k either remain in the complement of λ or remain in the same respective leaves during the entire isotopy. A *transverse measure* on λ is a function μ that associates to every arc in X transverse to λ a measure μ_k on k satisfying the following conditions:

- μ is invariant under transverse isotopies of arcs;
- if $k \subset k'$ then $\mu_k = \mu_{k'}|_k$;
- $\mathrm{supp}(\mu_k) = k \cap \lambda$.

A geodesic lamination is *compactly supported* if it is contained in a compact subset of X. A geodesic lamination is *measurable* if it is compactly supported and it admits a transverse measure. If two sublaminations of a geodesic lamination are both measurable then their union is also measurable. The largest measurable sublamination of a geodesic lamination is called its *stump*. The stump can possibly be empty.

3.3 Measured laminations

A *measured lamination* is a pair given by a compactly supported geodesic lamination and a transverse measure on it. The space of measured laminations on S is denoted by $\mathcal{ML}(S)$. It is a topological space homeomorphic to $\mathbb{R}^{6g-6+2p+3b}$ by [1, Proposition 3.9]. Two measured laminations are *projectively equivalent* if their underlying geodesic laminations coincide and their transverse measures differ by multiplication by a positive real number. A *projectivized measured lamination* is a projective equivalence class of non-trivial measured laminations. The space of projectivized measured laminations on S will be denoted by $\mathcal{PML}(S)$. It is homeomorphic to a sphere $\mathbb{S}^{6g-7+2p+3b}$ by [1, Proposition 3.9]. The simplest examples of measured laminations are given by the elements of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Each becomes a measured lamination once it is endowed with the *counting measure*, that is, the measure that counts the number of intersection points with a transverse arc.

The length function ℓ from Section 2.3 continuously extends to $\mathcal{ML}(S)$ (see [1]):

$$\ell : \mathrm{Teich}(S) \times \mathcal{ML}(S) \ni (X, \mu) \longrightarrow \ell_X(\mu) \in \mathbb{R}_+ .$$

The arc distance d_A in (2) can be also computed as follows.

Theorem 3.4 ([1, Proposition 3.3]). The following holds:

$$d_A(X, Y) = \max_{\mu \in \mathcal{ML}(S) \setminus \{\emptyset\}} \log \frac{\ell_Y(\mu)}{\ell_X(\mu)} = \max_{[\mu] \in \mathcal{PML}(S)} \log \frac{\ell_Y(\mu)}{\ell_X(\mu)} . \tag{6}$$

Note that the supremum over $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$ in (2) is now replaced by a maximum over $\mathcal{PML}(S)$ in (6). The measured laminations where the maximum is achieved are called *ratio-maximizing measured laminations* (see Section 9.2).

3.4 Generalities about Bonahon-Thurston's cocycles and cataclysms

In this section, we will recall some basics facts about the shearing cocycles of a hyperbolic structure, following Bonahon [2, 3]. This tool will be essential for us later, in Section 7.4.

In [2], Bonahon works for most of the paper under the hypothesis that the surface is closed. At the end, in Section 12.2 and 12.3, he explains how to extend his results to the case of a finite hyperbolic surface. Notice that in Bonahon's setting, laminations are not allowed to hit the boundary orthogonally, as we allow here. Hence, a maximal lamination decompose the surface into triangles.

For the rest of this section, we fix a maximal lamination λ on S that does not hit the boundary of S orthogonally. A *transverse cocycle* for λ can be thought of as a finitely additive signed measure for λ.

Definition 3.5 (Transverse cocycle). A *transverse cocycle* for λ is a map associating a number $\alpha(k) \in \mathbb{R}$ to each unoriented arc k transverse to λ such that α is *additive*, and α is λ-*invariant* (see [3] for more details). A transverse cocycle α satisfies the *cusp condition* if every simple closed curve transverse to λ and going once around a puncture of S has zero total measure for α. We denote by $H^0(\lambda;\mathbb{R})$ the set of all transverse cocycles for λ satisfying the cusp condition.

We will represent transverse cocycles using *train tracks*, see Bonahon [3, Section 3] for the definition. Following [3], a train track τ *snugly carries* a geodesic lamination λ if τ carries λ, if λ meets every tie of τ, and if there is no curve carried by τ which is disjoint from λ and which joins an endpoint of a spike of $S \setminus \tau$ to another one. Every train track which carries λ can be transformed into one that snugly carries λ after a finite sequence of splittings.

Definition 3.6 (Switch condition). Let τ be a train track, with set of edges E_τ. A function $\alpha : E_\tau \to \mathbb{R}$ is said to *satisfy the switch relations* if the following condition holds: for every switch v of τ, if $e_1, \dots, e_m$ are the edges arriving on one side of v and $f_1, \dots, f_n$ are the edges arriving on the other side, then

$$\sum_{i=1}^{m} \alpha(e_i) = \sum_{j=1}^{n} \alpha(f_i).$$

A function $\alpha : E_\tau \to \mathbb{R}$ is said to *satisfy the cusp condition* if for every puncture of S, the sum of the α-values of the edges of τ going into the puncture is zero.

Combining Theorem 11 and Theorem 17 in Bonahon's paper [3], we have the following:

Theorem 3.7 (Bonahon [3]). Let τ be a train track that snugly carries λ. There is a one-to-one correspondence between the set of all transverse cocycles for λ satisfying the cusp condition and the set of all the functions $\alpha : E_\tau \to \mathbb{R}$ which satisfy the switch relations and the cusp condition. In particular, the set $H^0(\lambda; \mathbb{R})$ is a finite dimensional vector space.

Every $X \in \mathrm{Teich}(S)$ induces a special transverse cocycle for λ, the *shearing cocycle*.

Proposition 3.8 (Bonahon [2]). Every $X \in \mathrm{Teich}(S)$ determines a unique transverse cocycle $\rho_X \in H^0(\lambda; \mathbb{R})$, called the *shearing cocycle* of X.

The previous proposition gives a natural map $\mathrm{Teich}(S) \to H^0(\lambda; \mathbb{R})$ defined as follows:

$$\mathrm{Teich}(S) \ni X \mapsto \rho_X \in H^0(\lambda; \mathbb{R})\,.$$

To understand this map, Bonahon uses *Thurston's symplectic form* ω on the vector space $H^0(\lambda, \mathbb{R})$. When λ is carried by a *generic* train track, that is, a train track where each switch is adjacent to exactly 3 edges, ω can be expressed by a simple formula. For generic train tracks, the three edges adjacent to the same switch play different roles: there is one edge on one side of the switch, called the *incoming edge*, and two edges on the other side, called *outgoing edges*. The two outgoing edges are called the *left* and the *right* outgoing edge, as seen from the incoming edge according to the orientation of S.

Lemma 3.9 ([18, §3.2], [2, Section 3]). If the train track is generic, ω can be expressed as

$$\omega(\alpha,\beta) = \sum_{v} [\alpha(e_v^r)\beta(e_v^l) - \alpha(e_v^l)\beta(e_v^r)]\,,$$

were the sum is taken over all the switches v of τ, e_v^l, e_v^r are the left and right outgoing edges from v, and $\alpha(e), \beta(e)$ are the weights associated to the edge e.

Theorem 3.10 (Bonahon [2]). A transverse cocycle $\alpha \in H^0(\lambda;\mathbb{R})$ is the shearing cocycle for a hyperbolic structure on S if and only if $\omega(\alpha,\beta) > 0$ for every compactly supported transverse measure β for λ. In particular, the map $X \mapsto \rho_X$ defines a real analytic homeomorphism from $\mathrm{Teich}(\Sigma)$ to an open convex cone bounded by finitely many faces in $H^0(\lambda;\mathbb{R})$.

Thurston's stretch lines can be easily described using this theory:

Proposition 3.11 (Bonahon [2]). For an $X \in \mathrm{Teich}(S)$, denote by $X_{Th}^t \in \mathrm{Teich}(S)$ Thurston's stretch line starting from X and directed by λ. Then,

$$\rho_{X_{Th}^t} = e^t \cdot \rho_X.$$

4 Average of Lipschitz Maps

In this section, we will deal with Lipschitz maps between convex hyperbolic surfaces. We introduce here a new technique that combines any two such maps into a new map whose Lipschitz constant is at most the average of their constants, generalizing a result by Guéritaud–Kassel [7]. We will employ this construction in Section 5.

4.1 The Guéritaud–Kassel construction for convex subsets of $\mathbb{H}^2$

Given two metric spaces (Ω, d) and (Ω', d'), a map $\phi : \Omega \to \Omega'$ is called $(K\text{-})$*Lipschitz* if there exists a real number $K \geq 0$ such that

$$d'(\phi(x_1), \phi(x_2)) \leq K d(x_1, x_2)$$

for all $x_1, x_2 \in \Omega$. The *Lipschitz constant* $\mathrm{Lip}(\phi)$ is the smallest of such K's.

The following criterion allows to recover the Lipschitz constant from local information.

Lemma 4.1 (Guéritaud–Kassel [7, Lemma 2.9]). Let Ω be a convex subset of $\mathbb{H}^2$ or a convex hyperbolic surface. Let (Ω', d') be a metric space. If $\phi : \Omega \to (\Omega', d')$ is a continuous function then we have the following:

$$\mathrm{Lip}(\phi) = \sup_{x \in \Omega} \inf_{r>0} \mathrm{Lip}(\phi_{|B(x;r)}).$$

We will be interested in a generalization of the following map.

Definition 4.2 (Guéritaud–Kassel [7, Section 2.3]). Let $\Omega, \Omega' \subseteq \mathbb{H}^2$ be convex subsets. Let $\phi, \psi : \Omega \to \Omega'$ be continuous maps. The *average* of ϕ and ψ is the map $\Upsilon : \Omega \to \Omega'$ such that for every $x \in \Omega$, $\Upsilon(x)$ is the midpoint of the geodesic joining $\phi(x)$ and $\psi(x)$.

Lemma 4.3 (Guéritaud–Kassel [7, Lemma 2.13]). Let $\Omega, \Omega' \subset \mathbb{H}^2$ be convex subsets, and $\phi, \psi : \Omega \to \Omega'$ be Lipschitz maps. Then their average $\Upsilon : \Omega \to \Omega'$ is a Lipschitz map and

$$\mathrm{Lip}(\Upsilon) \leq \frac{\mathrm{Lip}(\phi) + \mathrm{Lip}(\psi)}{2}.$$

4.2 Generalizing Guéritaud–Kassel's construction to convex hyperbolic surfaces

Assume that X, Y are two convex hyperbolic surfaces (not necessarily homeomorphic) and let $\phi, \psi : X \to Y$ be two continuous maps in the same homotopy class. We will now define a new continuous map $\Upsilon_{x_0,\gamma}(\phi, \psi) : X \to Y$ called the *average* of ϕ and ψ. This map generalizes the previous construction of Guéritaud–Kassel to hyperbolic surfaces. Our construction will depend on the choice of a base point $x_0 \in X$ and a geodesic path $\gamma : [0,1] \to Y$ joining $\phi(x_0)$ and $\psi(x_0)$.

Step 1: Construct two suitable lifts of ϕ and ψ

Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be the universal coverings of X and Y. Recall that $\widetilde{X}, \widetilde{Y} \subset \mathbb{H}^2$ are convex. Choose points $\widetilde{x_0} \in p^{-1}(x_0)$ and $\widetilde{y_0} \in q^{-1}(\phi(x_0))$. There exists a unique lift $\widetilde{\phi} : \widetilde{X} \to \widetilde{Y}$ of ϕ such that $\widetilde{\phi}(\widetilde{x_0}) = \widetilde{y_0}$. Similarly, there exists a unique lift $\widetilde{\gamma} : [0,1] \to \widetilde{Y}$ such $\widetilde{\gamma}(0) = \widetilde{y_0}$ (see the diagram below).

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\phi}"] \arrow[d, "p"'] & \widetilde{Y} \arrow[d, "q"] \\ X \arrow[r, "\phi"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{Y} \arrow[d, "q"] \\ {[0,1]} \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\psi}"] \arrow[d, "p"'] & \widetilde{Y} \arrow[d, "q"] \\ X \arrow[r, "\psi"'] & Y \end{tikzcd}$$

Denote by $\widetilde{z_0} = \widetilde{\gamma}(1)$. By construction $\widetilde{z_0} \in q^{-1}(\psi(x_0))$. There exists a unique lift $\widetilde{\psi} : \widetilde{X} \to \widetilde{Y}$ of ψ such that $\widetilde{\psi}(\widetilde{x_0}) = \widetilde{z_0}$. The construction above give us two maps $\widetilde{\phi}, \widetilde{\psi} : \widetilde{X} \to \widetilde{Y}$, between convex subsets of $\mathbb{H}^2$.

Step 2: Define $\Upsilon : \Omega \to \Omega'$
We can now define the map

$$\widetilde{\Upsilon} : \widetilde{X} \to \widetilde{Y}$$

that maps every $x \in \widetilde{X}$ to the midpoint of the geodesic segment joining $\widetilde{\phi}(x)$ and $\widetilde{\psi}(x)$ as in Definition 4.2. The lemma below follows easily from elementary arguments on coverings and their automorphisms.

Lemma 4.4. The map $\widetilde{\Upsilon} : \widetilde{X} \to \widetilde{Y}$ commutes with p and q.

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\Upsilon}"] \arrow[d, "p"'] & \widetilde{Y} \arrow[d, "q"] \\ X \arrow[r, "\Upsilon"'] & Y \end{tikzcd}$$

Definition 4.5. The (x_0, γ)-*average* between ϕ and ψ is the map $\Upsilon : X \to Y$ induced by $\widetilde{\Upsilon} : \tilde{X} \to \tilde{Y}$ and defined as follows:

$$\Upsilon(x) := q(\widetilde{\Upsilon}(\widetilde{x})),$$

where $\widetilde{x}$ is any element in $p^{-1}(x)$. We will also use the notation

$$\Upsilon_{x_0,\gamma}(\phi, \psi) := \Upsilon.$$

Using the basic properties of coverings, it is easy to verify that the map Υ does not depend on $\widetilde{y_0} \in q^{-1}(\phi(x_0))$ and $\widetilde{x_0} \in q^{-1}(x_0)$.

Theorem 4.6. Let X, Y be two (possibly non-homeomorphic) convex hyperbolic surfaces. Let $\phi, \psi : X \to Y$ be two homotopic Lipschitz maps. Then for every $x_0 \in X$ and for every $\gamma : [0, 1] \to Y$ with $\gamma(0) = \phi(x_0)$ and $\gamma(1) = \psi(x_0)$, the map $\Upsilon_{x_0,\gamma}(\phi, \psi) : X \to Y$

is Lipschitz with

$$\mathrm{Lip}(\Upsilon_{x_0,\gamma}(\phi,\psi)) \leq \frac{\mathrm{Lip}(\phi)+\mathrm{Lip}(\psi)}{2}.$$

Proof. We lift the maps ϕ and ψ to the universal coverings as in Step 1 above. The Lipschitz constant of $\widetilde{\Upsilon}$ is bounded by Lemma 4.3. This gives a bound on the Lipschitz constant of Υ by Lemma 4.1. ∎

5 Generalized Stretch Maps Between Geometric Pieces

We will construct optimal Lipschitz maps between geometric pieces of the same type.

Definition 5.1. Let $\mathcal{G}$ and $\mathcal{G}'$ be two geometric pieces of the same type. A continuous map $\phi : \mathcal{G} \to \mathcal{G}'$ is *label preserving* if it maps every edge of $\mathcal{G}$ to an edge of $\mathcal{G}'$ with the same label. Recall that the labels are assigned as in Fig. 1.

5.1 Centers and shears

Only triangular and quadrilateral pieces have bi-infinite edges. There is a one-parameter family of ways to glue two of them together along a bi-infinite edge. We will parametrize the glueing using the *shear parameter*, that is, the (signed) distance between their *centers*. We recall these key-definitions below.

Definition 5.2 (Center of l_i with respect to $\mathcal{T}$). Let l_i be a bi-infinite edge in a triangular piece $\mathcal{T}$. The *center* of l_i with respect to $\mathcal{T}$ is the intersection point $O^i_{\mathcal{T}}$ between l_i and the geodesic perpendicular to l_i through the opposite vertex.

Note that each triangular piece has three centers: $O^1_{\mathcal{T}}, O^2_{\mathcal{T}}, O^3_{\mathcal{T}}$.

Definition 5.3 (Center of l with respect to $\mathcal{Q}$). Let l be the (unique) bi-infinite edge in a quadrilateral piece $\mathcal{Q}$. The *center* of l with respect to $\mathcal{Q}$ is the intersection point $O_{\mathcal{Q}}$ between l and the unique (geodesic) perpendicular to l and to the opposite edge.

We are now in the position to define the following.

Definition 5.4 (Shear between $\mathcal{G}$ and $\mathcal{G}'$). Let $\mathcal{G}$ and $\mathcal{G}'$ be two geometric pieces glued along the bi-infinite edge e, as in Fig. 2. We define the *shear parameter* between $\mathcal{G}$ and $\mathcal{G}'$ as the signed distance $\mathrm{shear}_e(\mathcal{G},\mathcal{G}') \in \mathbb{R}$ between the centers $O_{\mathcal{G}}$ of e with respect to $\mathcal{G}$ and

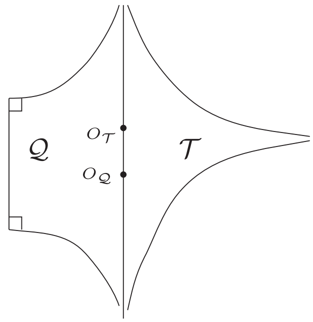


Fig. 2. Shear$(\mathcal{Q}, \mathcal{T}) > 0$.

the center $O_{\mathcal{G}'}$ of e with respect to $\mathcal{G}'$. The sign is given by the orientation of the surface, which we always assume to be counter-clockwise: the sign is positive if $O_\mathcal{G}$ comes before $O_{\mathcal{G}'}$ with respect to the orientation that $\partial\mathcal{G}$ induces on e. (Notice that flipping the roles of $\mathcal{G}$ and $\mathcal{G}'$ in this construction does not change the sign of $\text{shear}_e(\mathcal{G}, \mathcal{G}')$.)

5.2 Thurston's stretch homeomorphism between triangular pieces

Thurston [20] explicitly describes a family of Lipschitz homeomorphisms between ideal triangles.

Lemma 5.5 (Thurston [20, Proposition 2.2]). Let $t \geq 0$ and $\mathcal{T}, \mathcal{T}^t$ be two triangular pieces. There exists a label-preserving map $\phi^t : \mathcal{T} \to \mathcal{T}^t$ with the following properties:

1. ϕ is a homeomorphism;
2. $\phi^t(O_\mathcal{T}) = O_{\mathcal{T}^t}$ and its restriction $\phi^t_| : l_i \to l^t_i$ multiplies the arc length by e^t for each $i = 1, 2, 3$;
3. $\text{Lip}(\phi^t) = e^t$.

We denote the target of the map ϕ^t by $\mathcal{T}^t$ for consistency with the next sections ($\mathcal{T}^t$ is actually isometric to $\mathcal{T}$). In order to define his map ϕ^t, Thurston defined the *horocyclic foliation* a partial foliation $\mathcal{K}$ of $\mathcal{T}$ defined as follows (see Fig. 3). Consider the vertex of $\mathcal{T}$ adjacent to the edges l_1 and l_2. A horocycle h centered at this vertex intersects the edges l_1, l_2 at the points h_1, h_2, so that $d(h_1, O^1_\mathcal{T}) = d(h_2, O^2_\mathcal{T})$. We consider a partial foliation $\mathcal{K}_{12}$ whose leaves are all the horocycles whose points h_1, h_2 are closer to the vertex than the corresponding center $O^1_\mathcal{T}$ or $O^2_\mathcal{T}$. We denote by h^d_{12} the only horocycle h

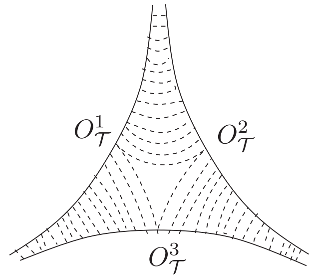


Fig. 3. Horocyclic foliation.

in $\mathcal{K}_{12}$ such that $d(h_1, O^1_{\mathcal{T}}) = d$. Similarly, we define partial foliations $\mathcal{K}_{23}, \mathcal{K}_{31}$ starting with the other two vertices.

Definition 5.6 (Horocyclic foliation). The *horocyclic foliation* $\mathcal{K}$ is the union of the three partial foliations $\mathcal{K}_{12}$, $\mathcal{K}_{23}$ and $\mathcal{K}_{31}$. (The triangle bounded by the three horocycles h^0_{12}, h^0_{23} and h^0_{31} is unfoliated.)

Lemma 5.7 (Thurston [20, Proposition 2.2]). The map ϕ^t maps the leaf h^d_{ij} of $\mathcal{K}$ in $\mathcal{T}$ to the leaf $h^{e^t d}_{ij}$ of $\mathcal{K}$ in $\mathcal{T}^t$ affinely. On the unfoliated region, ϕ^t is the identity.

In the rest of this section, we will prove results analog to Lemmas 5.5 and 5.7 for the geometric pieces of the other kinds.

5.3 Parameters for the geometric pieces

While there is just one ideal triangle up to isometry, the geometric pieces of the other kinds have parameters. A quadrilateral piece $\mathcal{Q}$ is uniquely determined by the length of the edge a_1, a pentagonal piece $\mathcal{P}$ by the lengths of the edges a_1, a_2, an hexagonal piece $\mathcal{H}$ by the lengths of three alternating edges, see Fig. 1. But these parametrizations are not very convenient for our needs. We will now introduce other parameters for the geometric pieces.

Doubling a quadrilateral piece $\mathcal{Q}$ along a_1, we get an ideal quadrilateral $\mathcal{Q}^d$, as in Fig. 4. Consider on a_1 the orientation induced by $\mathbb{H}^2$, and triangulate $\mathcal{Q}^d$ adding a diagonal e accordingly. Let $\mathcal{T}, \mathcal{T}'$ be the triangles obtained. Their shear s parametrizes $\mathcal{Q}$ completely. Notice that a_1 takes values in $(0, \infty)$, while s takes values in $(-\infty, \infty)$. As $s \to +\infty$ we have $a_1 \to +\infty$, but as $s \to -\infty$ we have $a_1 \to 0$.

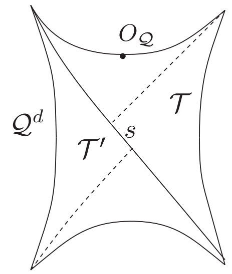


Fig. 4. The quadrilateral $\mathcal{Q}^d_s$.

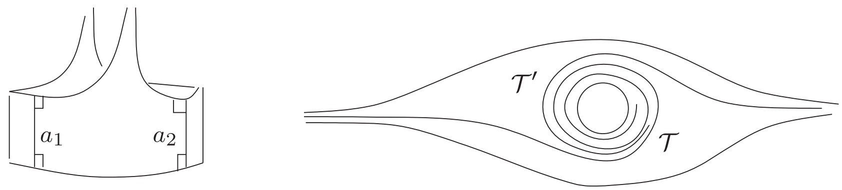


Fig. 5. Doubling the pentagon to get $\mathcal{P}^d$.

Similarly, doubling a pentagonal piece $\mathcal{P}$ along the edges a_1 and a_2, we get a hyperbolic cylinder $\mathcal{P}^d$ with two spikes and a totally geodesic boundary. We will denote it by l^d_1. After choosing an orientation on l^d_1, we consider two geodesics e_1, e_2 coming from each of the two spikes and spiraling around the geodesic l^d_1 according to the chosen orientation. The geodesics e_1, e_2 decompose $\mathcal{P}^d$ in two ideal triangles $\mathcal{T}, \mathcal{T}'$, as in Fig. 5. Their shears coordinates s_1 and s_2 parametrize $\mathcal{P}^d$ and hence $\mathcal{P}$: we will use them as parameters. The shear coordinates depend on the choice of orientation of the geodesic l^d_1. Indeed, by choosing the other orientation, we would have the mirror image of the same picture: the two shear coordinates would have the same absolute values but opposite signs. To fix the signs of the coordinates, we will always choose the orientation of l^d_1 so that $s_1 + s_2 > 0$. Note that this sum can never be zero as $|s_1 + s_2| = \ell(l^d_1)$ by [21, Proposition 3.4.21].

Again, doubling a hexagonal piece $\mathcal{H}$ along the edges a_1, a_2, a_3, we get a pair of pants $\mathcal{H}^d$ with 3 geodesic boundary components that we will denote by l^d_1, l^d_2, l^d_3. We can find three disjoint infinite simple geodesics e_1, e_2, e_3 in $\mathcal{H}^d$ such that e_i spirals around l^d_{i+1} and l^d_{i+2}, for $i \in \{1, 2, 3\}$ and sums of indices taken modulo 3. The geodesics e_1, e_2, e_3 cut the pair of pants in two triangles $\mathcal{T}, \mathcal{T}'$, glued with shear coordinates s_1, s_2, s_3. Up to changing the directions of the spirals, we can always choose the geodesics e_i such that $s_i + s_{i+1} = \ell(l^d_{i+2})$ (see [21, Proposition 3.4.21]). Thus, at least two of the s_i's are positive:

assume $s_2, s_3 > 0$. We will parametrize the hexagons using the three shear coordinates $s = (s_1, s_2, s_3)$.

In the three cases, the lengths of the edges a_i can be computed explicitly from the shears, but we will not need these formulae here.

5.4 Stretching the geometric pieces

In the rest of the paper, we will denote by $\mathcal{Q} := \mathcal{Q}_s, \mathcal{P} := \mathcal{P}_s$ with $s = (s_1, s_2)$, $\mathcal{H} := \mathcal{H}_s$ with $s = (s_1, s_2, s_3)$ a geometric piece having certain prescribed shear coordinates. Moreover, we will use the notation $\mathcal{Q}^t := \mathcal{Q}_{e^t s}, \mathcal{P}^t := \mathcal{P}_{e^t s}, \mathcal{H}^t := \mathcal{H}_{e^t s}$.

Lemma 5.8 (Stretch of geometric pieces). Fix $t \geq 0$. Let $\mathcal{F}_s = \mathcal{Q}_s, \mathcal{P}_s$ or $\mathcal{H}_s$ be a marked geometric piece. Then there exists a label-preserving map $\phi^t : \mathcal{F}_s \to \mathcal{F}_{e^t s}$ such that

1. ϕ^t is onto;
2. the map $\phi^t_| : l_i \to l_i^t$ is affine and multiplies the arc length by e^t for every $i = 1, 2, 3$;
3. $\mathrm{Lip}(\phi^t) = e^t$.
4. If the piece is a quadrilateral, $\phi^t(O_{\mathcal{Q}}) = O_{\mathcal{Q}^t}$;

Proof. We double $\mathcal{F}_s$ and $\mathcal{F}_{se^t}$ obtaining $\mathcal{F}_s^d$ and $\mathcal{F}_{e^t s}^d$, as we did in Section 5.3. Let $\sigma : \mathcal{F}_s^d \to \mathcal{F}_s^d$ be the isometric involution that maps one copy of $\mathcal{F}_s$ in $\mathcal{F}_s^d$ to the other copy. Similarly, let $\sigma^t : \mathcal{F}_{se^t}^d \to \mathcal{F}_{se^t}^d$ be the corresponding isometric involution on $\mathcal{F}_{se^t}^d$. Let $\mathcal{T}$ and $\mathcal{T}'$ be the two ideal triangles in $\mathcal{F}_s^d$ constructed in Section 5.3, separated by edges e_i (where $i = 1$ for $\mathcal{F}_s = \mathcal{Q}_s$, $i \in \{1, 2\}$ for $\mathcal{F}_s = \mathcal{P}_s$, $i \in \{1, 2, 3\}$ for $\mathcal{F}_s = \mathcal{P}_s$.) Let $\mathcal{T}_t$ and $\mathcal{T}_t'$ be the corresponding triangles in $\mathcal{F}_{se^t}^d$.

Let $\psi_t : \mathcal{T} \to \mathcal{T}_t$ and $\psi_t' : \mathcal{T}' \to \mathcal{T}_t'$ be the two homeomorphisms as in Lemma 5.5. The maps ψ_t and ψ_t' agree on the edges e_i, since $\mathrm{shear}_{e_i}(\mathcal{T}_t, \mathcal{T}_t') = e^t \cdot \mathrm{shear}_{e_i}(\mathcal{T}, \mathcal{T}')$. Hence, the maps ψ_t and ψ_t' glue to a homeomorphism $\Psi^t : \mathcal{F}_s^d \to \mathcal{F}_{se^t}^d$. By construction, Ψ^t maps every edge l_i of $\mathcal{F}_s^d$ to the corresponding edge of $\mathcal{F}_{se^t}^d$ multiplying its arc length by e^t.

By Lemma 4.1 we have

$$\mathrm{Lip}(\Psi^t) = \max\{\mathrm{Lip}(\psi_t), \mathrm{Lip}(\psi_t')\} = e^t.$$

Similarly, $\sigma_t \circ \Psi^t \circ \sigma : \mathcal{F}_s^d \to \mathcal{F}_{se^t}^d$ is a e^t-Lipschitz homeomorphism that maps every edge l_i of $\mathcal{F}_s^d$ to the corresponding edge of $\mathcal{F}_{se^t}^d$ multiplying its arc length by e^t.

Choose $x_0 \in a_1 \subset \mathcal{F}_s$. By construction we have $\sigma(x_0) = x_0$ and $\Psi^\sigma_t(x_0) = \sigma_t \circ \Psi^t(x_0)$. Let γ be a geodesic segment crossing a_1 once and joining $\Psi^t(x_0)$ and $\Psi^\sigma_t(x_0)$. Thus, γ is orthogonal to a_1 and $\sigma_t(\gamma) = \gamma$. We consider the map $\Upsilon^t := \Upsilon_{x_0,\gamma}(\Psi^t, \Psi^\sigma_t) : \mathcal{F}^d_s \to \mathcal{F}^d_{e^t s}$, that is, the average of Ψ^t and Ψ^σ_t with respect to x_0 and γ, in the sense of Theorem 4.6. By construction and Theorem 4.6, Υ^t enjoys the following properties:

- Υ^t is onto;
- Υ^t maps every edge l_i of $\mathcal{F}^d_s$ to the corresponding one of $\mathcal{F}_{se^t}$ by multiplying its arc length by e^t;
- $\mathrm{Lip}(\Upsilon^t) = e^t$.
- If $\mathcal{F}_s = \mathcal{Q}_s$, Υ^t maps the center of l_2 in $\mathcal{Q}^d_s$ to the center of l_2 in $\mathcal{Q}^d_{se^t}$;

Moreover, by construction we have

$$\sigma_t \circ \Upsilon^t \circ \sigma = \Upsilon_{\sigma(x_0),\sigma(\gamma)}(\sigma_t \circ \Psi \circ \sigma, \sigma_t \circ \Psi^\sigma \circ \sigma) = \Upsilon_{x_0,\gamma}(\Psi^\sigma, \Psi) = \Upsilon_{x_0,\gamma}(\Psi, \Psi^\sigma) = \Upsilon^t.$$

Hence, the image by Υ^t of the edges a_i of $\mathcal{F}_s$ are the corresponding edges of $\mathcal{F}_{e^t s}$, and Υ^t restricts to $\phi^t = \Upsilon_{|\mathcal{F}_s} : \mathcal{F}_s \to \mathcal{F}_{e^t s}$ as in the statement. ■

It is interesting to compare our construction of the stretch map for the case of the hexagon with the one given by Papadopoulos–Yamada [16], who construct optimal Lipschitz maps between special types of hexagons. Their work generalize an explicit example by Papadopoulos–Théret [15] for hexagons with $l_1 = l_2 = l_3$. The Lipschitz constant of the Papadopoulos–Yamada map is usually achieved only on one of the three alternating edges, but not on all of them. Because of this, their map is not suitable for our purposes.

5.5 Understanding the shear parameters

We will now give a geometric interpretation to the shear parameters s of the geometric pieces. Our aim is to prove Propositions 5.10, 5.12, and 5.14.

Consider $\mathcal{Q}_s := \overline{ABCD} \in \mathbb{H}^2$ where D, C are the two ideal vertices, as in Fig. 6. Let $\mathscr{F}$ be the horocyclic foliation based in C, with $f \in \mathscr{F}$ its (unique) leaf through $O_{\mathcal{Q}}$. Denote by P_{AD} the intersection point between f and the bi-infinite geodesic of $\mathbb{H}^2$ obtained extending $\overline{BC}$. Similarly, let $\mathscr{F}'$ be the horocyclic foliation based in D, with $f' \in \mathscr{F}'$ its (unique) leaf passing through $O_{\mathcal{Q}}$. Denote by P_{BC} the intersection point between f'

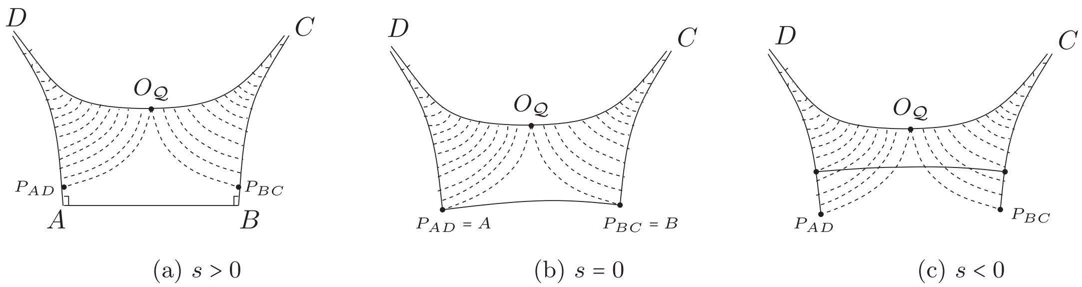


Fig. 6. Lemma 5.9.

and the bi-infinite geodesic of $\mathbb{H}^2$ obtained extending $\overline{AD}$. We will compute the "signed" distances between P_{BC}, P_{AB} and B, A, respectively. We define

1. $d_{\pm}(P_{BC}, B) := \epsilon \cdot d(P_{BC}, B)$, where $\epsilon = 1$ when $P_{BC} \in \overline{BC}$ and $\epsilon = -1$ when $P_{BC} \notin \overline{BC}$;
2. $d_{\pm}(P_{AD}, A) := \epsilon \cdot d(P_{AD}, A)$, where $\epsilon = 1$ when $P_{AD} \in \overline{AD}$ and $\epsilon = -1$ when $P_{AD} \notin \overline{AD}$.

By construction, it is clear that $d_{\pm}(P_{BC}, B) = d_{\pm}(P_{AD}, A)$.

Lemma 5.9. In the notation above, we have

$$d_{\pm}(P_{BC}, B) = d_{\pm}(P_{AD}, A) = \frac{s}{2}.$$

Proof. We will compute these lengths explicitly. We will denote by C^u, D^u the vertices of $\mathcal{Q}_s^d$ which are the reflection of C, D. The ideal quadrilateral $\mathcal{Q}_s^d$ can be drawn in the upper half plane model of $\mathbb{H}^2$, with vertices $D^u = -1, C^u = 0, C = e^s, D = \infty$, see Fig. 7. With this choice, the two ideal triangles $\mathcal{T} = \overline{C^u CD}$ and $\mathcal{T}' = \overline{D^u C^u D}$ are glued with shear coordinate equal to s.

We will first compute the coordinates of the center $O_{\mathcal{Q}} \in \overline{CD}$. We denote by $O_{\mathcal{Q}}^u \in \overline{C^u D^u}$ the reflection of $O_{\mathcal{Q}}$. The geodesic segment $\overline{O_{\mathcal{Q}} O_{\mathcal{Q}}^u}$ is the common perpendicular of the geodesics $\overline{CD}$ and $\overline{C^u D^u}$. In the language of Euclidean geometry, $\overline{O_{\mathcal{Q}} O_{\mathcal{Q}}^u}$ is an arc of a Euclidean circle centered at C and perpendicular to $\overline{C^u D^u}$. By a computation, the Euclidean radius of this circle is $\sqrt{e^s(1+e^s)}$. This number is also the y-coordinate of the points $O_{\mathcal{Q}}$ and P_{AD} (see Fig. 7).

We will now compute the coordinates of the point A in a similar way. The geodesic segment $\overline{AB}$ is the common perpendicular of the geodesics $\overline{CC^u}$ and $\overline{DD^u}$. In the language of Euclidean geometry, $\overline{AB}$ is an arc of a Euclidean circle centered at D^u and

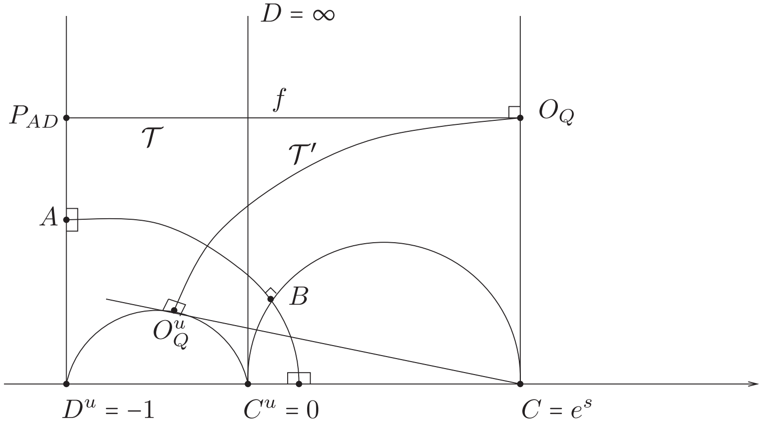


Fig. 7. The quadrilateral $\mathcal{Q}^d$ in the upper half plane model.

perpendicular to $\overline{CC^u}$. By a computation, the Euclidean radius of this circle is $\sqrt{1+e^s}$. This is also the y-coordinate of A.

The number $d_{\pm}(P_{AD}, A)$ is the log of the ratio of the y-coordinates of P_{AD} and A:

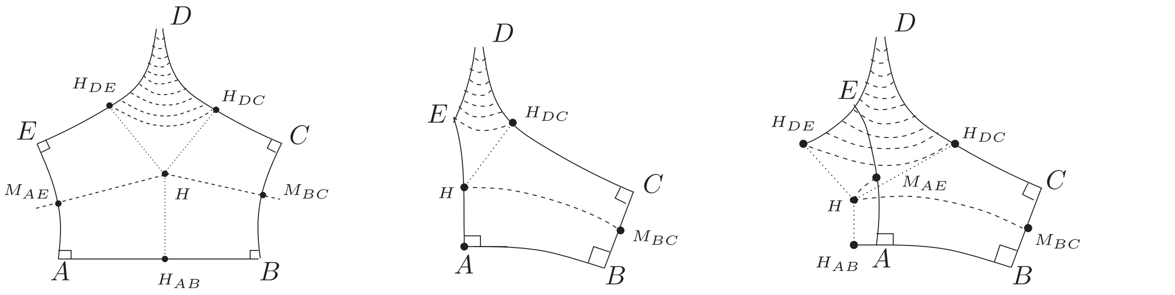
$$d_{\pm}(P_{AD}, A) = \log \frac{\sqrt{e^s(1+e^s)}}{\sqrt{1+e^s}} = \frac{s}{2}\,.$$

■

The following result is an immediate consequence of Lemmas 5.8 and 5.9

Proposition 5.10. Let $t \geq 0$, and let $\phi^t : \mathcal{Q} \to \mathcal{Q}^t$ be the generalized stretch map as in Lemma 5.8. Then, if $s \geq 0$, the map ϕ^t sends the points P_{BC} and P_{AD} of $\mathcal{Q}$ to the points P_{BC} and P_{AD}, respectively, of $\mathcal{Q}^t$.

Let $s = (s_1, s_2)$ with $s_1 + s_2 > 0$. Up to changing the order of s_1 and s_2, we can assume that $s_2 > s_1$, which in particular gives $s_2 > 0$. Consider the pentagon $\mathcal{P}_s := \overline{ABCDE} \in \mathbb{H}^2$ where D is the ideal vertex, as in Fig. 8. The axes of the segments $\overline{AE}$ and $\overline{BC}$ intersect in a point H, which can be inside $\mathcal{P}$, outside $\mathcal{P}$ or on the side $\overline{EA}$, see Fig. 8. We will see in Lemma 5.11 that this depends on the sign of s_1. Notice that the point H lies on the bisector of the ideal angle at the vertex D. Let M_{AE}, M_{BC} be the midpoints of $\overline{AE}, \overline{BC}$, respectively, and H_{AB}, H_{DC}, H_{DE} the projections of H on the geodesics containing $\overline{AB}, \overline{DC}, \overline{DE}$, respectively. Denote $\mathscr{F}$ be the horocyclic foliation based in D. By construction there is one unique leaf $f \in \mathscr{F}$ passing through H_{DE} and H_{DC}.



(a) $s_1, s_2 > 0$ (b) $s_1 = 0, s_2 > 0$ (c) $s_1 < 0, s_2 > 0$

Fig. 8. Lemma 5.11.

We compute the signed distances between E and H_{DE}, and B and H_{AB}. We define

1. $d_{\pm}(H_{DE}, E) := \epsilon \cdot d(H_{DE}, E)$, where $\epsilon = 1$ when $H_{DE} \in \overline{ED}$ and $\epsilon = -1$ when $H_{DE} \notin \overline{ED}$;
2. $d_{\pm}(H_{DC}, C) := \epsilon \cdot d(H_{DC}, C)$, where $\epsilon = 1$ when $H_{DC} \in \overline{DC}$ and $\epsilon = -1$ when $H_{DC} \notin \overline{DC}$;
3. $d_{\pm}(H_{AB}, A) := \epsilon \cdot d(H_{AB}, A)$, where $\epsilon = 1$ when $H_{AB} \in \overline{AB}$ and $\epsilon = -1$ when $H_{AB} \notin \overline{AB}$;
4. $d_{\pm}(H_{AB}, B) := \epsilon \cdot d(H_{AB}, B)$, where $\epsilon = 1$ when $H_{AB} \in \overline{AB}$ and $\epsilon = -1$ when $H_{AB} \notin \overline{AB}$.

By construction, it follows $d_{\pm}(H_{DE}, E) = d_{\pm}(H_{AB}, A)$ and $d_{\pm}(H_{DC}, C) = d_{\pm}(H_{AB}, B)$.

Lemma 5.11. In the notation above, we have

$$d_{\pm}(H_{DE}, E) = d_{\pm}(H_{AB}, A) = \frac{s_1}{2}$$
$$d_{\pm}(H_{DC}, C) = d_{\pm}(H_{AB}, B) = \frac{s_2}{2}.$$

Proof. We will compute these lengths explicitly. Denote by D^u the spike of $\mathcal{P}_s^d$ which is the reflection of D. The universal covering of $\mathcal{P}_s^d$ can be drawn in the upper half plane model of $\mathbb{H}^2$. We will denote by $\widetilde{D}$ a lift of D, by $\widetilde{D^u_+}$ the lift of D^u at its left and by $\widetilde{D^u_-}$ the lift of D^u at its right, see Fig. 9. We remark that the left part of Fig. 9 is drawn in the disc model for an easier visualization, but all the computations are performed in the upper half plane model. Denote by $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{E}$ the lifts of A, B, C, E, which form a copy of the pentagon with the vertex $\widetilde{D}$. Denote by Z and W the endpoints of the lift of the geodesic l_1^d.

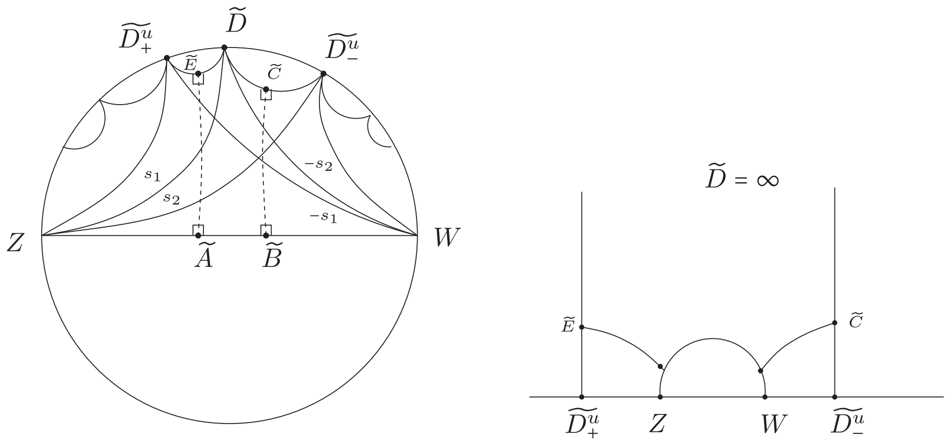


Fig. 9. The universal covering of $\mathcal{P}^d$.

We can assume $Z = 0$, $\widetilde{D^u_+} = -1, \widetilde{D} = \infty$. Using the two triangles $\mathcal{T}$ and $\mathcal{T}'$, glued with shears s_1, s_2, we find $\widetilde{D^u_-} = e^{s_1}$. Similarly, using the ideal triangulation whose triangles spiral around l^d_1 in the opposite direction, glued with shears $-s_1, -s_2$, we find

$$W = \frac{e^{s_1+s_2} - 1}{e^{s_2} + 1} \ .$$

We will now compute the coordinates of the points $\widetilde{A}$ and $\widetilde{E}$. The geodesic containing them is perpendicular to the lift of l^d_1, hence it lies on an Euclidean circle centered at $\widetilde{D^u_+}$. By an elementary computation its Euclidean radius is $r = \sqrt{\frac{e^{s_2}(e^{s_1}+1)}{e^{s_2}+1}}$. Hence, the point $\widetilde{E}$ is the complex number $-1 + ri$. The point $\widetilde{A}$ can be found as the intersection of two circles:

$$\widetilde{A} = \frac{e^{s_1+s_2} - 1}{e^{s_1+s_2} + 2e^{s_2} + 1}(1 + ir).$$

We now compute the intersection between the axis of the geodesic containing $\widetilde{A}$ and $\widetilde{E}$ and the bisector of the ideal angle at $\widetilde{D}$. The bisector is the vertical line with real part equal to $\frac{e^{s_1}-1}{2}$:

$$\left\{ \frac{e^{s_1} - 1}{2} + it \mid t > 0 \right\}.$$

To compute its intersection with the axis, we will apply the Möbius transformation:

$$M : z \to \frac{-z-1-r}{z+1-r} \, .$$

The transformation acts in the following way:

$$M\left(\widetilde{E}\right) = i, \quad M\left(\widetilde{A}\right) = i\frac{1+2r+e^{s_2}(2+e^{s_1}+2r)}{e^{s_1+s_2}-1} \, .$$

The axis of the segment between $M\left(\widetilde{E}\right)$ and $M\left(\widetilde{A}\right)$ is given by the equation

$$\left\{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \text{ and } z\bar{z} = \frac{1+2r+e^{s_2}(2+e^{s_1}+2r)}{e^{s_1+s_2}-1} \right\} \, .$$

The imaginary part of the intersection between the bisector and the axis is given by the equation:

$$M\left(\frac{e^{s_1}-1}{2}+it\right)\overline{M\left(\frac{e^{s_1}-1}{2}+it\right)} = \frac{1+2r+e^{s_2}(2+e^{s_1}+2r)}{e^{s_1+s_2}-1} \, ,$$

with solution

$$t = \frac{1}{2}\sqrt{\frac{(e^{s_1}+1)(3e^{s_1+s_2}-e^{s_1}-e^{s_2}-1)}{(e^{s_2}+1)}} \, .$$

This number is the imaginary part of the point $\widetilde{H}$, whose real part is $\frac{e^{s_1}-1}{2}$.

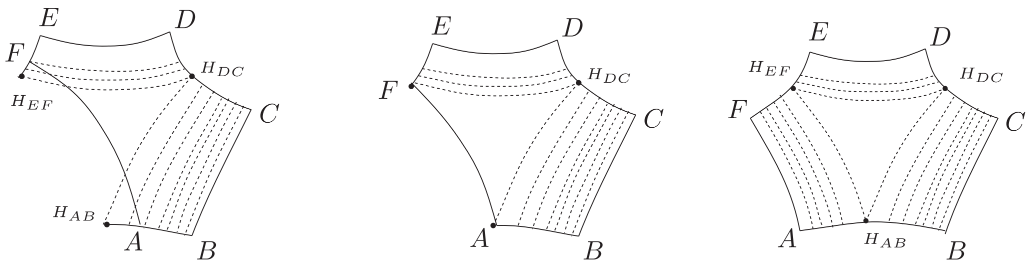
The point H_{DE} lies on the perpendicular line from $\widetilde{H}$ to the segment from $\widetilde{A}$ to $\widetilde{E}$. This line lies on a Euclidean circle centered at $\widetilde{D^u_+}$, hence the imaginary part of H_{DE} is equal to the radius of this circle, which is the absolute value of $\widetilde{H}+1$, namely

$$\sqrt{\frac{e^{s_1+s_2}(e^{s_1}+1)}{e^{s_2}+1}}.$$

The number $d_\pm(H_{DE}, E)$ is the log of the ratio of the imaginary parts of $\widetilde{H_{DE}}$ and $\widetilde{E}$:

$$d_\pm(H_{DE}, E) = \log \frac{\sqrt{\frac{e^{s_1+s_2}(e^{s_1}+1)}{e^{s_2}+1}}}{\sqrt{\frac{e^{s_2}(e^{s_1}+1)}{e^{s_2}+1}}} = \frac{s_1}{2} \, .$$

For $d_\pm(H_{DC}, C)$, notice that $d_\pm(H_{DE}, E) + d_\pm(H_{DC}, C) = l_1 = \frac{1}{2}(s_1+s_2)$. ■



(a) $s_1 < 0, s_2, s_3 > 0$ (b) $s_1 = 0, s_2, s_3 > 0$ (c) $s_1, s_2, s_3 > 0$

Fig. 10. Lemma 5.13.

The following result is an immediate consequence of Lemmas 5.8 and 5.11.

Proposition 5.12. Let $t \geq 0$, and let $\phi^t : \mathcal{P} \to \mathcal{P}^t$ be the generalized stretch map as in Lemma 5.8. Then, the map ϕ^t sends the point H_{DC} of $\mathcal{P}$ to the point H_{DC} of $\mathcal{P}^t$. Moreover, if $s_1 \geq 0$, ϕ^t sends the points H_{DE} and H_{AB} of $\mathcal{P}$ to the points H_{DE} and H_{AB}, respectively, of $\mathcal{P}^t$.

Let $s = (s_1, s_2, s_3)$. Consider the hexagon $\mathcal{H}_s := \overline{ABCDEF} \subset \mathbb{H}^2$ as in Fig. 10: l_1 is the edge $\overline{CD}$, l_2 is the edge $\overline{AB}$, and l_3 is the edge $\overline{EF}$. Consider the axes of the segments $\overline{BC}$, $\overline{DE}$, $\overline{FA}$. The three axes all meet in a common point H. Let H_{AB}, H_{DC}, H_{EF} be the orthogonal projections of H on the geodesics containing the segments $\overline{AB}$, $\overline{DC}$ and $\overline{EF}$ (see also [16]).

Consider two consecutive vertices V, W of the hexagon, the orthogonal projection H_{VW} of H on the geodesic $\overline{VW}$. We define the signed distance of H_{VW} from V:

$$d_{\pm}(H_{VW}, V) := \epsilon \cdot d(H_{VW}, V),$$

where $\epsilon = 1$ if H_{VW} lies on the geodesic ray starting from V that contains $\overline{VW}$, and $\epsilon = -1$ if H_{VW} lies on the geodesic ray starting from V that does not contain $\overline{VW}$.

Lemma 5.13. We have

$$\begin{aligned} d_{\pm}(H_{EF}, F) = d_{\pm}(H_{AB}, A) &= \frac{s_1}{2}, \\ d_{\pm}(H_{DC}, C) = d_{\pm}(H_{AB}, B) &= \frac{s_2}{2}, \\ d_{\pm}(H_{EF}, E) = d_{\pm}(H_{DC}, D) &= \frac{s_3}{2}. \end{aligned}$$

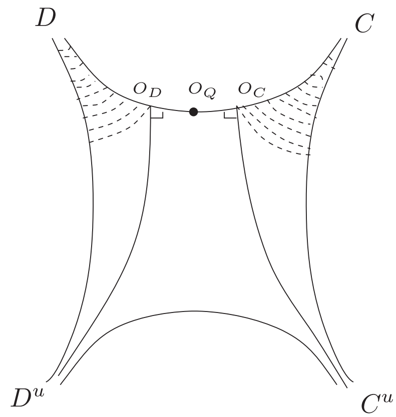


Fig. 11. Horocyclic foliation.

Proof. The equalities between signed distances come from the construction of H (see Papadopoulos–Yamada [16]). Using these equalities, we find a linear system, which admits a unique solution, see Fig. 10:

$$d_{\pm}(H_{DC}, D) + d_{\pm}(H_{AB}, B) = \ell(l_1)\,,$$

$$d_{\pm}(H_{AB}, B) + d_{\pm}(H_{EF}, F) = \ell(l_2)\,,$$

$$d_{\pm}(H_{EF}, F) + d_{\pm}(H_{DC}, D) = \ell(l_3)\,.$$

Since by our initial assumptions we have $\frac{s_i}{2} + \frac{s_{i+1}}{2} = \ell(l_{i+2})$ for $i = 1, 2, 3$, we conclude. ∎

The following result is an immediate consequence of Lemmas 5.8 and 5.13.

Proposition 5.14. Let $t \geq 0$ and let $\phi^t : \mathcal{H} \to \mathcal{H}^t$ be the generalized stretch map as in Lemma 5.8. Then, the map ϕ^t sends the point H_{DC} of $\mathcal{H}$ to the point H_{DC} of $\mathcal{H}^t$. Moreover, if $s_1 \geq 0$, ϕ^t sends the points H_{DE} and H_{AB} of $\mathcal{H}$ to the points H_{DE} and H_{AB}, respectively, of $\mathcal{H}^t$.

5.6 The horocyclic foliation

We will now construct a partial foliation $\mathcal{K}$, called the *horocyclic foliation*, in every quadrilateral and pentagonal piece, see Fig. 11. Our aim is to prove Propositions 5.16 and 5.19.

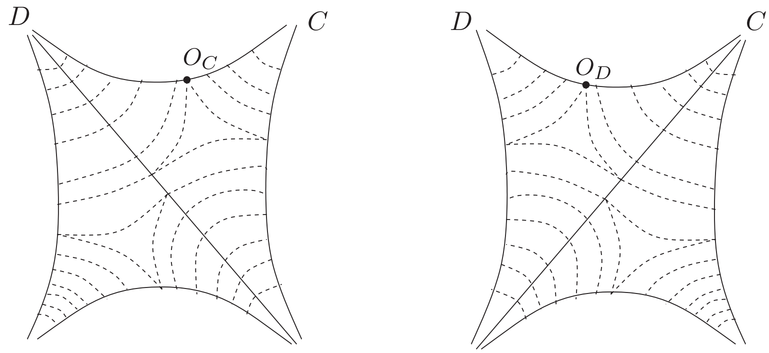


Fig. 12. Lemma 5.16.

For a quadrilateral piece $\mathcal{Q}$, see Fig. 11. Recall that C^u, D^u denote the vertices of $\mathcal{Q}_s^d$ which are the reflection of C, D. Denote by O_C, O_D the points of $\overline{CD}$ that are the nearest point projections of C^u, D^u, respectively. We consider a partial foliation $\mathcal{K}_C$ whose leaves are all the horocycles centered at C, which intersect the side $\overline{CD}$ between C and O_C. Similar definition for a partial foliation $\mathcal{K}_D$.

Definition 5.15 (Horocyclic foliation). The *horocyclic foliation* $\mathcal{K}$ is the union of the two partial foliations $\mathcal{K}_C$ and $\mathcal{K}_D$.

From the computations in the proof of Lemma 5.9 we have

$$d(O_{\mathcal{Q}}, O_C) = d(O_{\mathcal{Q}}, O_D) = \tfrac{1}{2}\log(1 + e^{-s}).$$

For $d \geq \frac{1}{2}\log(1 + e^{-s})$, we denote by h_C^d the only horocycle in $\mathcal{K}_C$ where the distance between $O_{\mathcal{Q}}$ and its intersection with the edge $\overline{CD}$ equals d.

Lemma 5.16. The map ϕ^t in Lemma 5.8 maps the leaf h_C^d of $\mathcal{K}$ in $\mathcal{Q}$ to the leaf $h_C^{e^t d}$ of $\mathcal{K}$ in $\mathcal{Q}^t$ affinely. Similarly, ϕ^t maps the leaf h_D^d of $\mathcal{K}$ in $\mathcal{Q}$ to the leaf $h_D^{e^t d}$ of $\mathcal{K}$ in $\mathcal{Q}^t$ affinely.

Proof. The map ϕ^t is the average of the two maps Ψ^t and $\sigma^t \circ \Psi^t \circ \sigma$ (see the proof of Lemma 5.8). The map Ψ^t is represented on the left-hand side of Fig. 12, the map $\sigma^t \circ \Psi^t \circ \sigma$ on the right-hand side. Each map sends the horocycle h_C^d, with $d \geq \frac{1}{2}\log(1 + e^{-s})$, to a horocycle, hence their average will also send this horocycle to a horocycle, which must be $h_C^{e^t d}$ by part (2) and (3) of Lemma 5.8. This horocycle is still in $\mathcal{K}$ for $\mathcal{Q}^t$, because a simple computation shows that if $d \geq \frac{1}{2}\log(1 + e^{-s})$ then $e^t d \geq \frac{1}{2}\log(1 + e^{-e^t s})$.

■

Now, let $\mathcal{P}$ be a pentagonal piece. Using the notation as in the proof of Lemma 5.11, denote by O_W the point of $\overline{\widetilde{D}\widetilde{E}}$ that is the nearest point projection of W.

Definition 5.17 (Horocyclic foliation). We define the *horocyclic foliation* $\mathcal{K}$ as the partial foliation whose leaves are all the horocycles centered at $\widetilde{D}$, which intersect $\overline{\widetilde{D}\widetilde{E}}$ between $\widetilde{D}$ and O_W.

Lemma 5.18. The point O_W lies on the half-line $\overline{\widetilde{D}\widetilde{E}}$, and we have

$$d(O_W, \widetilde{E}) = \frac{s_2}{2} + \tfrac{1}{2} \log\left(\frac{e^{s_1}+1}{e^{s_2}+1}\right).$$

Proof. From the computation in the proof of Lemma 5.11, we have

$$\mathrm{Im}(O_W) = W + 1 = r^2 .$$

Since $r > 1$, we see that O_W is above $\widetilde{E}$ and their distance is

$$d(O_W, \widetilde{E}) = \log(r) = \tfrac{1}{2} \log\left(\frac{e^{s_2}(e^{s_1}+1)}{e^{s_2}+1}\right).$$

■

For $d \geq \frac{s_2}{2} + \frac{1}{2}\log\left(\frac{e^{s_1}+1}{e^{s_2}+1}\right)$, we denote by h^d the only horocycle in $\mathcal{K}$ where the distance between O_W and its intersection with the edge $\overline{\widetilde{D}\widetilde{E}}$ equals d.

Lemma 5.19. The map ϕ^t from Lemma 5.8 maps the leaf h^d of $\mathcal{K}$ in $\mathcal{P}$ to the leaf $h^{e^t d}$ of $\mathcal{K}$ in $\mathcal{P}^t$ affinely.

Proof. The map ϕ^t is the average of the two maps Ψ^t and $\sigma^t \circ \Psi^t \circ \sigma$ (see the proof of Lemma 5.8). Each of the two maps sends the horocycle h^d, with $d \geq \frac{s_2}{2} + \frac{1}{2}\log\left(\frac{e^{s_1}+1}{e^{s_2}+1}\right)$, to a horocycle, hence their average will also send this horocycle to a horocycle, which must be $h^{e^t d}$ by parts (2) and (3) of Lemma 5.8. This horocycle is still in $\mathcal{K}$ for $\mathcal{P}^t$, because a computation shows that $d \geq \frac{s_2}{2} + \frac{1}{2}\log\left(\frac{e^{s_1}+1}{e^{s_2}+1}\right)$ implies

$$e^t d \geq \frac{e^t s_2}{2} + \tfrac{1}{2}\log\left(\frac{e^{e^t s_1}+1}{e^{e^t s_2}+1}\right).$$

■

Remark 5.20. It might seem more natural to extend the horocyclic foliation $\mathcal{K}$ on a quadrilateral piece $\mathcal{Q}$ until the point $O_{\mathcal{Q}}$ as in Fig. 6. Similarly, for a pentagonal piece $\mathcal{P}$, it might seem more natural to extend it until the point H_{DE} as in Fig. 8. Unfortunately ϕ^t does not map all the horocycles of these extended foliations to horocycles. This property only holds for the leaves of $\mathcal{K}$.

6 The Boundary Block

Let $X \in \mathrm{Teich}(S)$ and λ a maximal lamination on X. In this section and the following, we construct some auxiliary surfaces that we will use to define our generalized stretch lines. Here we define the boundary block of λ in X, that is the subset of X obtained as the union of all the geometric pieces that are not ideal triangles. It comes equipped with a finite maximal lamination λ_B, consisting of the boundary leaves of these pieces. The boundary block is non-empty if and only if at least one of the leaves of λ is orthogonal to the boundary of X. After the definition of the boundary block, we will describe how to "stretch" it using the results of Section 5.

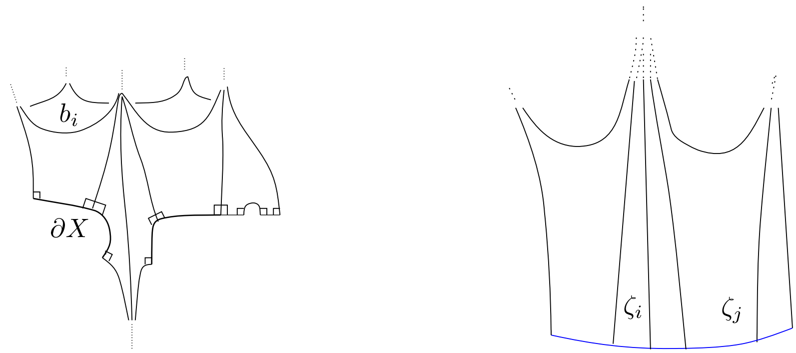
6.1 Definition of the boundary block

We define the *boundary block* of λ as the subset $B \subset X$ obtained as a union of all the geometric pieces of $X \setminus \lambda$ that have at least one edge on ∂X, that is, quadrilaterals, pentagons, and hexagons:

$$B := \bigcup \{\mathcal{G}_i \mid \mathcal{G}_i \text{ is a geometric piece of } X \setminus \lambda \text{ of type (2), (3) or (4) } \} \subset X\,.$$

By construction B is a (possibly disconnected) 2-manifold with boundary. Notice that its boundary in general might not be compact. The inclusion map $B \hookrightarrow X$ induces via pull-back a Riemannian metric on B, which turns B into a (possibly disconnected) complete hyperbolic surface of finite volume. Hence, the connected components of B are convex hyperbolic surfaces. Notice that the inclusion map $B \hookrightarrow X$ is a 1-1 local isometry, but not necessarily an isometric embedding in the sense of metric spaces. Indeed, the infinite geodesics in the boundary of the quadrilateral and pentagonal pieces might spiral in a bounded region of X, but they are not contained in a bounded region for the hyperbolic metric on B.

The boundary of B contains compact and non-compact components. We will denote by $\partial^c B$ the union of the compact components of ∂B, and by $\partial^{nc} B$ the union of



(a) A cycle $c \subset \partial^{nc} B$ (b) The crown $C \subset B$ of c

Fig. 13. A cycle $c \subset \partial^{nc} B$ and its associated crown $C \subset B$.

the non-compact components of ∂B:

$$\partial B = \partial^c B \cup \partial^{nc} B.$$

The compact boundary components are also boundary components of X. The non-compact boundary components are bi-infinite geodesics that are boundary of quadrilaterals. Every non-compact boundary component of B has two ideal vertices corresponding to two spikes.

Definition 6.1 (Cycle in $\partial^{nc} B$). A *cycle* in $\partial^{nc} B$ is a cyclically ordered set $c := \{b_1, \dots, b_s\}$ of components of $\partial^{nc} B$ such that for every i the geodesic b_i shares a spike with b_{i-1} and with b_{i+1}, and b_s shares a spike with b_1. We will denote by Q_i the quadrilateral piece containing b_i in its boundary. We will also denote by a_i the spike shared by b_i and b_{i+1} and by a_s the spike shared by b_s and b_1 (see Fig. 13).

The boundary block B has finitely many cycles $c_1, \dots, c_m$ in $\partial^{nc} B$. Every cycle c_i in $\partial^{nc} B$ determines a (unique) simple closed geodesic γ_i in its homotopy class.

Definition 6.2 (Crown spanned by a cycle). The *crown* spanned by c_i is the subsurface $C_i := \mathrm{ConvHull}(c_i, \gamma_i) \subset B$ which is the convex hull of c_i and γ_i. By construction C_i is a complete hyperbolic surface whose interior is topologically a cylinder, and $\partial C_i = \gamma_i \cup c_i$ (see Fig. 13).

Lemma 6.3. If $c_j, c_k \in \partial^{nc} B$ are distinct cycles, then C_j and C_k are disjoint.

Notation 6.4. We will use the following notation:

- $C := \bigcup_i C_i$, where C_i is the crown associated to the cycle $c_i \subset \partial^{nc} B$;
- $B_C := \overline{B \setminus C} \subset B \subset X$ is the closure of the complement of C in B;
- $\Gamma = \bigcup \gamma_i$ is the finite union of all the pairwise disjoint simple closed geodesics γ_i;
- $X_C := \overline{X \setminus B_C} \subset X$ is the complement of B_C in X.

The following proposition is an immediate consequence of our constructions:

Proposition 6.5. The maximal lamination λ induces a decomposition of X as follows:

$$X = X_C \cup B \text{ with } X_C \cap B = C\,,$$

where X_C is a (possibly disconnected) finite hyperbolic surface with the metric induced by X and

$$\partial X_C = \Gamma \cup (\partial X \setminus \partial B)\ .$$

Furthermore, all of the following holds:

- $\lambda_B := \{l \in \lambda \mid l \text{ is a leaf entirely contained in } B\}$ is a maximal lamination for B;
- $\lambda_{X_C} := \{l \in \lambda \mid l \text{ is a leaf entirely contained in } X_C\}$ is a lamination for X_C;
- $\lambda_B \cap \lambda_{X_C}$ is the union of the non-compact boundary components of B;
- $\lambda = \lambda_B \cup \lambda_{X_C}$.

Notice that by definition the following holds:

- λ_{X_C} does not contain any leaf that hits the boundary of X perpendicularly;
- $X_C \setminus \lambda_{X_C} = \mathring{C} \cup \bigcup \{\mathring{\mathcal{G}} \mid \mathcal{G} \text{ is a triangular geometric piece of } X \setminus \lambda\}\,.$

6.2 Stretching the boundary block

For every $t \geq 0$ we will now stretch the boundary block along λ_B. We will denote the new hyperbolic surface by $B^t_{\lambda_B}$ or simply B^t.

As a 1st step, for every geometric piece $\mathcal{G}_i$ involved in the definition of B consider its stretched analog $\mathcal{G}_i^t$ defined in Section 5. We define B^t to be the (possibly disconnected) surface obtained by glueing together these $\mathcal{G}_i^t$'s following the glueing pattern of the corresponding $\mathcal{G}_i$'s in B:

$$B^t := \bigcup \{\mathcal{G}_i^t \mid \mathcal{G}_i \text{ is a geometric piece of } X \setminus \lambda \text{ of type (2), (3), or (4) } \}.$$

The edges of the pieces are glued pairwise via isometries according to the following rules:

- When two pieces are glued along edges of finite length, our definition of the $\mathcal{G}_i^t$ guarantees that the corresponding edges have the same length, hence there is only one way to glue them by an isometry;
- When two pieces are glued along half-infinite edges, again there is only one way to glue them by an isometry and making the common vertices coincide (for the quadrilateral pieces, the half-infinite edges are AD or BC in Fig. 6, for the pentagonal pieces the half-infinite edges are AD or CD in Fig. 8);
- Two pieces are glued together along a bi-infinite edge if and only if both are quadrilaterals. If s_0 is the shear between $\mathcal{G}_1$ and $\mathcal{G}_2$ for the surface B according to Definition 5.4, we glue $\mathcal{G}_1^t$ and $\mathcal{G}_2^t$ by an isometry with shear $e^t \cdot s_0$.

Notice that all the glueings are well defined. As a consequence, we have the following:

Lemma 6.6. B^t is a (possibly disconnected) complete hyperbolic surface of finite volume diffeomorphic to B.

By construction every cycle of bi-infinite leaves $c_i \subset \partial^{nc}B$ corresponds to a cycle $c_{i^t} \subset \partial^{nc}B^t$. Every c_{i^t} determines a simple closed geodesic γ_{i^t} and a crown

$$C_{i^t} := \mathrm{ConvHull}(c_{i^t}, \gamma_{i^t}) \subset B^t \sim .$$

The C_{i^t}'s are all disjoint, we denote their (disjoint) union by

$$C^t := \bigcup C_{i^t} \subset B^t \sim .$$

Proposition 6.7 (Existence of a stretch map for B). For every $t \geq 0$ there exists a continuous map $\beta^t : B \to B^t$ homotopic to the identity with the following properties:

1. $\beta^t(\partial B) = \partial B^t$;
2. β^t stretches the arc length of the leaves of λ_B by e^t;

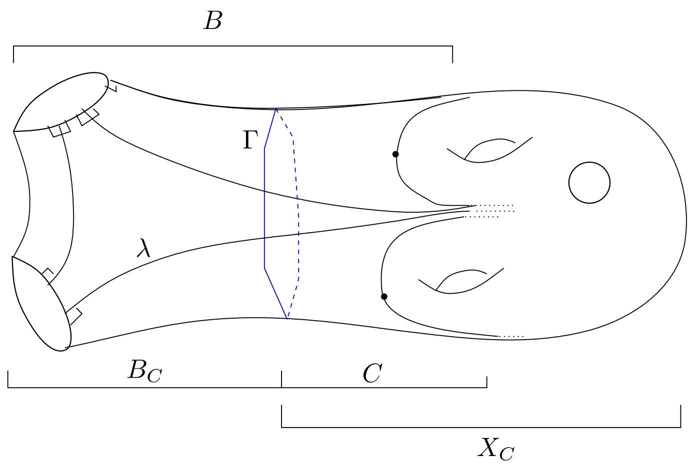


Fig. 14. The λ-decomposition of X when C is connected and $\Gamma = \{\gamma\}$.

3. on every geometric piece $\mathcal{G}$ in $B \setminus \lambda_B$ the map β^t restricts to $\beta^t|_{\mathcal{G}} = \phi^t : \mathcal{G} \to \mathcal{G}^t$ as in Lemma 5.8;
4. $\mathrm{Lip}(\beta^t) = e^t$.

Proof. We define β^t by glueing together the maps $\phi^t : \mathcal{G}_i \to \mathcal{G}_i^t$ as in Lemma 5.8 following the glueings of the $\mathcal{G}_i^t$'s in B^t. We have that $\mathrm{Lip}(\beta^t) \leq e^t$ by Lemma 4.1, and $\mathrm{Lip}(\beta^t) \geq e^t$ because β^t stretches the arc length of the leaves of λ_B by e^t. ■

7 The Triangulated Surface

Here we will construct our 2nd auxiliary surface. Again, let $X \in \mathrm{Teich}(S)$ and λ be a maximal lamination on X. In Section 6.1, we defined the subsurface X_C endowed with the lamination λ_{X_C}. Notice that λ_{X_C} is not maximal for X_C: it divides X_C into triangles and the crowns C_j (see Fig. 14). We will define a new complete hyperbolic surface: the *triangulated surface* X_A. This surface will extend X_C, that is, it will come equipped with an isometry $g : X_C \hookrightarrow X_A$.

The triangulated surface is not embedded in X, it will be constructed by suitably gluing new triangles to the crowns C_j of X_C. The surface X_A will also be equipped with a maximal lamination $\lambda_A \supset \lambda_{X_C}$ such that $g(\lambda \cap X_C) \subset \lambda_A \cap X_C$ and $X_A \setminus \lambda_A$ is a union of ideal triangles (Fig. 15).

We will then stretch the triangulated surface X_A by using Bonahon's theory of cataclysms (see Section 3.4), which uses a cocycle to deform a triangulated surface.

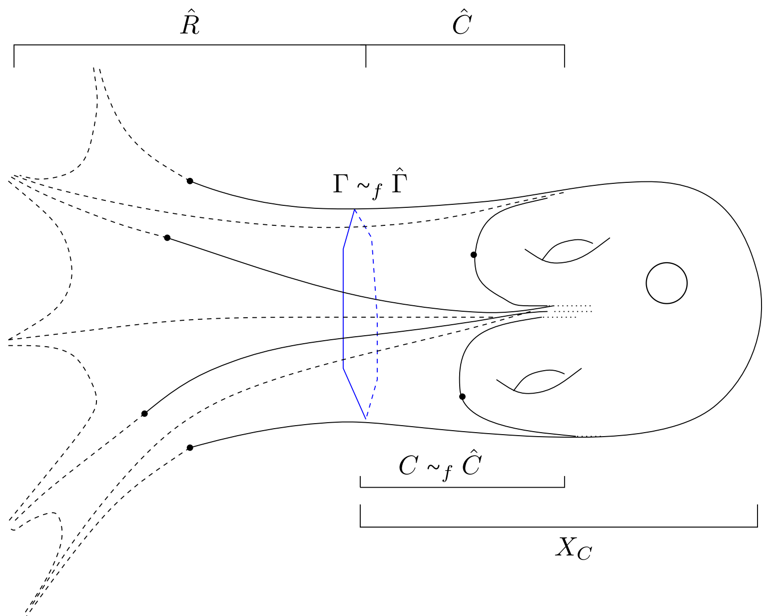


Fig. 15. The surface X_A.

We will first stretch the auxiliary cylinders in Section 7.3, and finally we will define the stretched triangulated surface, by constructing a suitable cocycle in Section 7.4. The stretched boundary block and the stretched triangulated surface will be key ingredients in the construction of the generalized stretch lines discussed in Section 8.

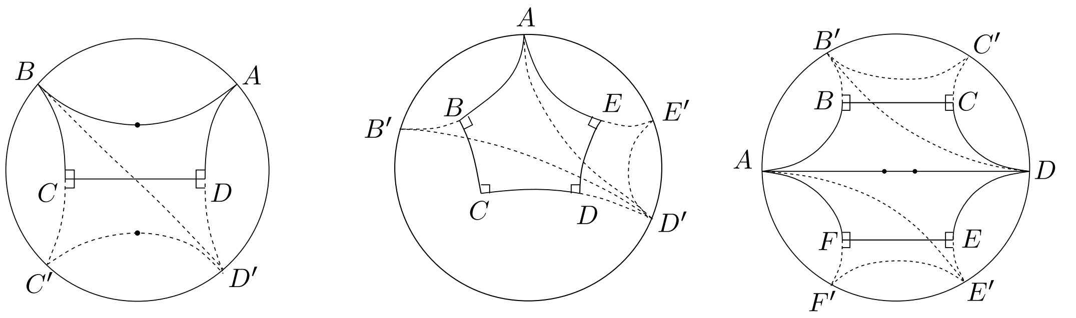
7.1 Extension of a crown

As a 1st step in the construction of X_A, we will first work with crowns. Consider a crown $C_j \subset X_C$. We will extend it to a complete hyperbolic surface A_j, homeomorphic to an annulus, equipped with an isometry

$$f_j : C_j \hookrightarrow A_j$$

and a finite maximal lamination δ_{A_j} with $f_i(\lambda \cap C_j) \subset \delta_{A_j} \cap C_j$.

For every crown C_j and every geometric piece $\mathcal{G}_k$, we define the region $\zeta_k^j := \mathcal{G}_k \cap C_j$, see Fig. 13. Now each crown is decomposed as $C_j = \bigcup_k \zeta_k^j$. For every region ζ_k^j, we will now define a triangulated ideal polygon $\widehat{\mathcal{G}_k}$ with an isometry $\zeta_k^j \hookrightarrow \widehat{\mathcal{G}_k}$. After this, we will define A_j by replacing each ζ_k^j with $\widehat{\mathcal{G}_k}$.



(a) $\widehat{\mathcal{Q}} := ABC'D'$ (b) $\widehat{\mathcal{P}} := AB'D'E'$ (c) $\widehat{\mathcal{Q}} \cup \widehat{\mathcal{Q}'}$

Fig. 16. Construction of $\widehat{\mathcal{G}_k}$.

We will consider three cases. Every region ζ_k^j has one or two spikes: if it has two spikes, then $\mathcal{G}_k$ is a quadrilateral, see Case (1) below; if it has one spike there are two possibilities, either $\mathcal{G}_k$ is a pentagon, as in Case (2) below, or $\mathcal{G}_k$ is again a quadrilateral and one of the adjacent pieces is again a quadrilateral glued to $\mathcal{G}_k$ along a bi-infinite edge, see Case (3) below.

Defining $\widehat{\mathcal{G}_k}$: Case (1) We assume that ζ_k^j has two spikes and $\zeta_k^j \subset \mathcal{G}_k = \mathcal{Q}$ is a quadrilateral, denoted by $\mathcal{Q} := \overline{ABCD} \subset \mathbb{H}^2$, see Fig. 16. We orient the finite edge $\overline{CD}$ from C to D, according to the orientation of $\mathcal{Q}$. We define $\widehat{\mathcal{G}_k}$ as the double of $\mathcal{Q}$ along $\overline{CD}$, as in Fig. 16:

$$\widehat{\mathcal{G}_k} = \mathcal{Q}^d := \overline{ABC'D'}\,.$$

This ideal quadrilateral will be triangulated by adding the diagonal $\overline{BD'}$. The diagonal can be chosen in a consistent way using the orientation of $\overline{CD}$. We will call the edges $\overline{BC'}$ and $\overline{AD'}$ *special edges*, and the points C and D *special points*.

Defining $\widehat{\mathcal{G}_k}$: Case (2) We assume that ζ_k^j has one spike and $\zeta_k^j \subset \mathcal{G}_k = \mathcal{P}$ is a pentagon, denoted by $\mathcal{P} := \overline{ABCDE} \subset \mathbb{H}^2$ as in Fig. 16. We orient the edge $\overline{CD}$ of $\mathcal{P}$ opposite to the spike from C to D, following the orientation of $\mathcal{P}$. We define $D', B', E' \in \partial\mathbb{H}^2$ as the extremes of the geodesics $\overline{CD}, \overline{AB}, \overline{AE}$ on the side of D, B, E, respectively. We define

$$\widehat{\mathcal{G}_k} := \overline{AB'D'E'}\,.$$

This ideal quadrilateral will be triangulated by adding the diagonal $\overline{AD'}$. We will call the edges $\overline{AB'}$ and $\overline{AE'}$ *special edges*, and the points B and E *special points*.

Defining $\widehat{\mathcal{G}_k}$: Case (3) We assume that ζ^j_k, ζ^j_{k+1} both have one spike and $\zeta^j_k \subset \mathcal{G}_k = \mathcal{Q}$, $\zeta^j_{k+1} \subset \mathcal{G}_{k+1} = \mathcal{Q}'$, where both $\mathcal{Q}, \mathcal{Q}'$ are quadrilaterals, denoted by $\mathcal{Q} := \overline{ABCD}, \mathcal{Q}' := \overline{AFED}$, where $\overline{AD}$ is the bi-infinite edge they share, see Fig. 16. We define $B', C', E', F' \in \partial\mathbb{H}^2$ as the endpoints of the geodesics $\overline{AB}, \overline{DC}, \overline{DE}, \overline{AF}$ on the side of B, C, E, F, respectively.

If the common spike of ζ^j_k and ζ^j_{k+1} is D, we define

$$\widehat{\mathcal{G}_k} := \overline{ADC'B'} \text{ and } \widehat{\mathcal{G}_{k+1}} := \overline{ADE'}.$$

The ideal quadrilateral $\widehat{\mathcal{G}_k}$ will be triangulated by adding the diagonal $\overline{B'D}$. We will call the edges $\overline{AD}$, $\overline{C'D}$ its *special edges*, and the points $O_{\mathcal{Q}}$ and C its *special points*. For the ideal triangle $\widehat{\mathcal{G}_{k+1}}$, we will call the edges $\overline{AD}$, $\overline{DE'}$ its *special edges* and the points $O_{\mathcal{Q}'}$ and E its *special points*.

If the common spike of ζ^j_k and ζ^j_{k+1} is A, we define

$$\widehat{\mathcal{G}_k} := \overline{AB'D} \text{ and } \widehat{\mathcal{G}_{k+1}} := \overline{AF'E'D}.$$

For the ideal triangle $\widehat{\mathcal{G}_k}$ we will call the edges $\overline{AD}$, $\overline{AB'}$ its *special edges* and the points $O_{\mathcal{Q}}$ and B its *special points*. The ideal quadrilateral $\widehat{\mathcal{G}_{k+1}}$ will be triangulated by adding the diagonal $\overline{AE'}$. We will call the edges $\overline{AD}$, $\overline{AF'}$ its *special edges*, and the points $O'_{\mathcal{Q}}$ and F its *special points*.

Definition of the auxiliary cylinder In all cases above, we have an isometry $\zeta^j_k \hookrightarrow \widehat{\mathcal{G}_k}$. Now we define A_j glueing together the $\widehat{\mathcal{G}_k}$'s according to the glueing pattern of the associated $\zeta^j_k \subset \mathcal{G}_k$:

$$A_j := \bigcup \left\{ \widehat{\mathcal{G}_k} \mid \widehat{\mathcal{G}_k} \text{ is the triangulated ideal polygon tailored to } \mathcal{G}_k \supset \zeta^j_k \text{ as above} \right\} / \sim,$$

where $\widehat{\mathcal{G}_h}$ and $\widehat{\mathcal{G}_{h'}}$ are glued together if and only if ζ^j_h and $\zeta^j_{h'}$ are adjacent in C_j or, equivalently, if and only if their associated geometric pieces $\mathcal{G}_h$ and $\mathcal{G}_{h'}$ are adjacent in B. More precisely, consider two consecutive geometric pieces $\mathcal{G}_h, \mathcal{G}_{h+1}$, and denote by $e_h \subset \mathcal{G}_h$ and $e_{h+1} \subset \mathcal{G}_{h+1}$ the two edges that are glued together in B. We denote by

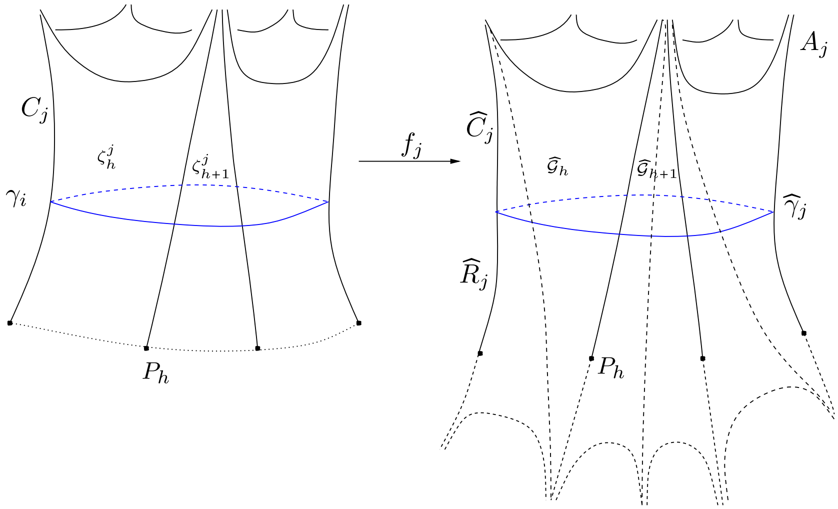


Fig. 17. $A_j = \widehat{C}_j \cup \widehat{R}_j$ with $\widehat{C}_j \cap \widehat{R}_j = \widehat{\gamma}_j$, $f_j : C_j \to \widehat{C}_j \subset A_j$ and $P_h \sim P_{h+1}$.

$\widehat{e}_h \subset \widehat{\mathcal{G}_h}$ and $\widehat{e}_{h+1} \subset \widehat{\mathcal{G}_{h+1}}$ the corresponding special edges of $\widehat{\mathcal{G}_h}, \widehat{\mathcal{G}_{h+1}}$, and by $\widehat{P_h} \in \widehat{e_h}$ and $\widehat{P_{h+1}} \in \widehat{e_{h+1}}$ their special points. There are two cases:

- if e_h, e_{h+1} are half-infinite edges, glue together $\widehat{e}_h, \widehat{e}_{h+1}$ with an isometry that makes the special points $\widehat{P_h}$ and $\widehat{P_{h+1}}$ coincide (see Fig. 17).
- (this can happen only in the Case (3)) if e_h, e_{h+1} are bi-infinite edges, the glueing procedure of $\widehat{\mathcal{G}_h}$ and $\widehat{\mathcal{G}_{h+1}}$ is the one described in Case (3) above. The edges $\widehat{e}_h, \widehat{e}_{h+1}$ are glued together with an isometry that keeps the special points $\widehat{P_h}$ and $\widehat{P_{h+1}}$ at a distance equal to the shear between the quadrilateral pieces $\mathcal{G}_h, \mathcal{G}_{h+1}$.

In this way, for every crown $C_j \subset C$, we constructed a space A_j that we call the *auxiliary cylinder* of C_j. We remark that our construction relies only on the choice of an orientation on X. It satisfies the following properties:

Lemma 7.1. For every C_j, the surface A_j constructed above is a complete hyperbolic surface whose interior is homeomorphic to an annulus. Moreover, there exists an isometry $f_j : C_j \hookrightarrow A_j$.

Proof. By construction A_j is an annulus and $\partial A_j = \{c_j', c_j''\}$ where c_j', c_j'' are cycles of bi-infinite geodesics, with the leaves of c_j' in bijection with leaves of $c_j \in \partial C_j$. Moreover, by construction, there exists $\epsilon > 0$ and an isometry $f_\epsilon : N_\epsilon(c_j) \hookrightarrow N_\epsilon(c_j')$. Denote by $\hat{\gamma}_j$

the core geodesic of the annulus and $\widehat{C}_j := \mathrm{ConvHull}(c'_j, \hat{\gamma}_j) \subset A_j$. We have that $\widehat{C}_j$ is a complete hyperbolic surface. Moreover f_ϵ extends to an isometry $f_j : C_j \hookrightarrow \widehat{C}_j$. ■

We will denote by δ_{A_j} the (finite) maximal lamination of A_j given by the union of all the edges of the ideal triangles in the triangulations of the $\widehat{\mathcal{G}}_k$'s. This lamination has a natural partition, which will be useful later: we will denote by δ'_{A_j} the union of all the special edges of the $\widehat{\mathcal{G}}_k$'s by δ''_{A_j} is the union of all the extra diagonals we added to the $\widehat{\mathcal{G}}_k$'s.

7.2 The triangulated surface

We define the *auxiliary multi-cylinder* A as the disjoint union of all the auxiliary cylinders A_j. There is an isometry $f : C \hookrightarrow A$ defined as $f(z) := f_j(z)$ if $z \in C_j$. This multi-cylinder carries a finite maximal lamination δ_A defined as the union of all the δ_{A_j}. This is again partitioned in two parts: $\delta'_A := \bigcup_j \delta'_{A_j}$ consisting of the special edges, $\delta''_A := \bigcup_j \delta''_{A_j}$ made of the extra diagonals.

Notation 7.2. The following notation will be used here and in the rest of the paper:

- $\widehat{C}_j := f_j(C_j)$, $\widehat{C} := f(C) = \bigsqcup \widehat{C}_j$;
- $\hat{\gamma}_j := f_j(\gamma_j)$, $\widehat{\Gamma} := f(\Gamma) = \bigsqcup \widehat{\gamma}_j$;
- $\widehat{R}_j := \overline{A_j \setminus \widehat{C}_j} = \overline{A_j \setminus f_j(C_j)}$, $\widehat{R} := \overline{A \setminus \widehat{C}} = \overline{A \setminus f(C)} = \bigsqcup \widehat{R}_j$.

Thus, in this notation we have $A = \widehat{C} \cup \widehat{R}$ with $\widehat{R} \cap \widehat{C} = \widehat{\Gamma}$ and $f : C \to \widehat{C}$ is an isometry (see Fig. 17). As above, $\widehat{R}$ is a complete hyperbolic surface.

Definition 7.3 (The triangulated surface). We define a surface X_A, called the *triangulated surface* as follows (see also Fig. 15):

$$X_A := X_C \bigsqcup A \sim, \text{where } z \sim f(z) \text{ for every } z \in C.$$

Let $\pi : X_C \bigsqcup A \to X_A$ be the quotient map associated. We denote by $g := \pi_| : A \to X_A$ the restriction of π to A, and by λ_A the lamination

$$\lambda_A := \pi(\lambda_{X_C}) \cup \pi(\delta_A) \subset X_A.$$

We denote by μ_A the closure of $\pi(\delta_A)$ in X_A, a sublamination of λ_A. We denote by ν_A the lamination $\mu_A \setminus \pi(\delta_A)$, a sublamination of λ_A. Notice that

$$\nu_A \subset \mu_A \subset \lambda_A.$$

Proposition 7.4. The quotient map $\pi : X_C \bigsqcup A \to X_A$ induces on X_A a structure of (possibly disconnected) complete hyperbolic surface of finite volume with non-compact boundary. The following diagrams are commutative and all arrows are 1-1 local isometries:

$$\begin{tikzcd} & X_C \arrow[dr, "\pi_|"] & \\ C \arrow[ur, "\iota"] \arrow[dr, "f"'] & & X_A \\ & A \arrow[ur, "g:=\pi_|"'] & \end{tikzcd} \qquad \begin{tikzcd} & X_C \arrow[dr, "\pi_|"] & \\ \Gamma \arrow[ur, "\iota_|"] \arrow[dr, "f_|"'] & & X_A \\ & A \arrow[ur, "g:=\pi_|"'] & \end{tikzcd}$$

where $\iota : C \hookrightarrow X_C$ is the canonical inclusion; $g := \pi_| : A \to X_A$ and $\pi_| : X_C \to X_A$ are the restrictions of π. Moreover, $\lambda_A := \pi(\lambda_{X_C}) \cup \pi(\delta_A)$ is a maximal lamination on X_A, and $\pi_| : X_C \to \overline{X_A \setminus g(\widehat{R})}$ is an isometry.

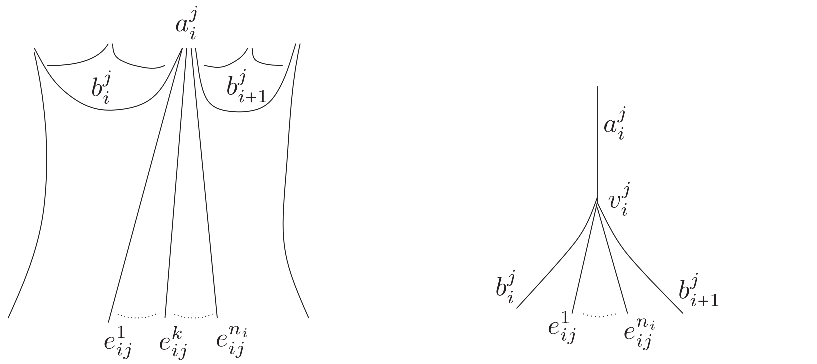
Proof. Notice that, by our constructions, X_A can be equivalently defined as follows:

$$X_A = X_C \bigsqcup \widehat{R} \sim, \quad \text{where } z \sim f(z) \text{ for every } z \in \Gamma.$$

So X_A is obtained glueing together two complete hyperbolic surfaces (X_C and $\widehat{R}$) along finitely many compact connected components of their boundary ($\Gamma \subset \partial X_C$ and $\widehat{\Gamma} \subset \partial \widehat{R}$) via a prescribed isometry ($f_| : \Gamma \to \widehat{\Gamma}$). Therefore, X_A is also a (possibly disconnected) complete hyperbolic surface, and the two restrictions $\pi_{|A}$, $\pi_{|X_C}$ of π are both 1-1 local isometries. The following holds by our definitions of B, X_C and λ_{X_C}:

$$X_C \setminus \lambda_{X_C} = \mathring{C} \cup \{\mathring{\mathcal{G}} \mid \mathcal{G} \text{ is a triangular geometric piece in } X \setminus \lambda\}\,.$$

By construction $A \setminus \delta_A$ is a union of ideal triangles. Since π identifies C with $\widehat{C} \subset A$, we thus have that $X_A \setminus \lambda_A$ is a union of ideal triangles as well, that is, λ_A is maximal. ■



(a) The spike a_i^j of A_j

(b) Its corresponding subtrack $\tau_{ij} \subset \tau$

Fig. 18. Notation on A_j.

7.3 Stretching the auxiliary cylinders

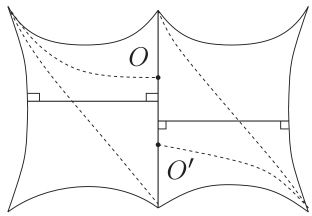
We now want to stretch the triangulated surface X_A. We will start by stretching the auxiliary cylinders. In Section 7.1 we defined the auxiliary cylinder A_j for every crown C_j. In Section 6.2 we defined the stretched boundary block B^t and we introduced the crown C_j^t in B^t. We now want to define the stretched auxiliary cylinder A_j^t for every parameter $t \geq 0$.

Definition 7.5 (Stretched auxiliary cylinder). We define the *stretched auxiliary cylinder* A_j^t as the auxiliary cylinder associated to the crown C_j^t, that is, we apply the definition of Section 7.1 to the crown C_j^t. By construction we also get an isometry $f_j^t : C_j^t \hookrightarrow A_j^t$ and a maximal finite lamination δ_{A_j} on A_j^t. By Lemma 7.1 A_j^t is a complete hyperbolic surface whose interior is homeomorphic to an annulus.

Notation 7.6. We denote by $c_j := \bigcup_i b_i^j \subset \partial A_j$ the cycle of bi-infinite leaves that corresponds to the cycle with the same name in B. We say:

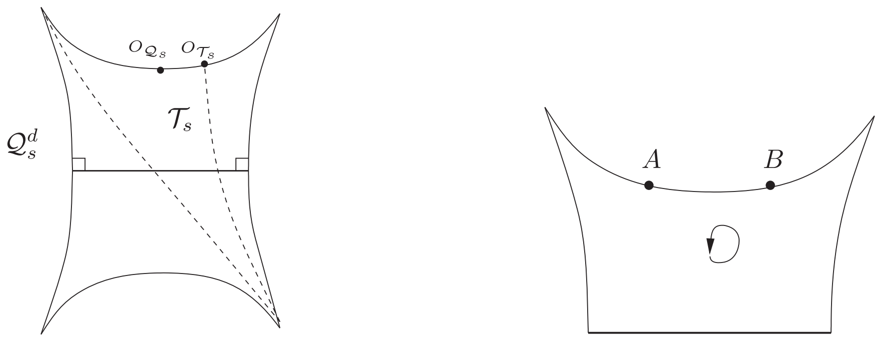
- b_i^j is the bi-infinite leaf in $\partial \widehat{\mathcal{Q}_i} \cap \partial A_j$ with $\mathcal{Q}_i := \mathcal{Q}_{s_i} \subset B$ a quadrilateral piece. The b_i^j's are enumerated so that any two consecutive b_i^j, b_{i+1}^j form a spike $a_i^j \subset A_j$;
- the leaves of δ_{A_j} entering the spike a_i^j are denoted by $e_{ij}^1, \ldots, e_{ij}^{n_i}$ (Fig. 18). Note that $\delta_{A_j} = \bigcup_{a_i^j} \{e_{ij}^1, \ldots, e_{ij}^{n_i}\}$.

The lamination δ_{A_j} on A_j is actually an ideal triangulation. We associate to each edge the shear between the two adjacent triangles (see Definition 5.4).



(a) $(A_j)^t_{\mathrm{Th}}$ (b) A^t_j

Fig. 19. Comparing A^t_j with $(A_j)^t_{\mathrm{Th}}$.



(a) $\delta(s) := O_{\mathcal{T}_s} - O_{\mathcal{Q}_s}$ (b) Notation: $A - B > 0$

Fig. 20. The displacement function δ.

Definition 7.7 (Shear coordinates for A_j and A^t_j). We denote the shear coordinates of the hyperbolic structure A_j by $s^0(e^k_{ij})$, and the shear coordinates of A^t_j by $s^t(e^k_{ij})$.

If $e^k_{ij} \in \delta_{A_j}$ is not a special edge then $s^t(e^k_{ij}) = e^t \cdot s^0(e^k_{ij})$ by construction of A^t_j. Otherwise, in general $s^t(e^k_{ij}) \neq e^t \cdot s^0(e^k_{ij})$ (see Fig. 19 and Proposition 7.9).

7.3.1 *Stretch difference formula*

The auxiliary cylinder A_j is triangulated by δ_{A_j}, so it can be stretched using Thurston's technique [20]. We denote by $(A_j)^t_{\mathrm{Th}}$ the Thurston's stretch of A_j. The shear coordinates of $(A_j)^t_{\mathrm{Th}}$ are $e^t \cdot s^0(e^k_{ij})$ by construction. In Proposition 7.9 we quantify the difference between the shear coordinates of A^t_j and $(A_j)^t_{\mathrm{Th}}$.

Notation 7.8. In this section and in the proof of Lemma 8.3, if A, B are points on the bi-infinite edge of a quadrilateral piece, we will denote by $A - B$ their signed distance, with the sign given by the orientation of the surface (Fig. 20).

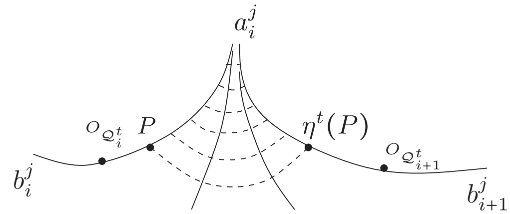


Fig. 21. The horocycle map $\eta^t : b_i^j \to b_{i+1}^j$.

Definition 7.9 (Displacement function). Let $\mathcal{Q}_s$ be the quadrilateral piece of shear $s \in \mathbb{R}$ and $\mathcal{T}_s \subset \mathcal{Q}_s^d$ be the ideal triangle adjacent to the bi-infinite edge $b \subset \mathcal{Q}_s$. Let $O_{\mathcal{Q}_s}$ be the center of $\mathcal{Q}_s$ and $O_{\mathcal{T}_s}$ the center of $\mathcal{T}_s$ on the edge b (see Fig. 20). We define the *displacement function* $\delta : \mathbb{R} \to \mathbb{R}^+$ as follows:

$$\delta(s) := O_{\mathcal{T}_s} - O_{\mathcal{Q}_s}.$$

(Note that the function $\delta : \mathbb{R} \to \mathbb{R}^+$ is continuous and bijective.)

Definition 7.10 (Horocyclic map). For every spike $a_i^j \in A_j^t$ consider the *horocyclic map*

$$\eta^t : b_i^j \to b_{i+1}^j,$$

where $\eta^t(P) \in b_{i+1}^j$ is the endpoint of the (unique) horocycle around the spike a_i^j through $P \in b_i^j$ (Fig. 21).

By definition of the shear coordinates on A_j^t we have

$$\sum_{k=1}^{n_i} s^t(e_{ij}^k) = \eta^t(O_{\mathcal{T}_i}) - O_{\mathcal{T}_{i+1}^t}\ . \tag{7}$$

Lemma 7.11. For every $t \geq 0$ we have $\eta^t(O_{\mathcal{Q}_i^t}) - O_{\mathcal{Q}_{i+1}^t} = e^t \cdot [\eta^0(O_{\mathcal{Q}_i^0}) - O_{\mathcal{Q}_{i+1}^0}]$.

Proof. See Fig. 21. Let h^t be the horocycle around the spike a_i^j passing through $O_{\mathcal{Q}_i^t}$. We denote by $e_{ij}^{k_0}, e_{ij}^{k_1}, \dots, e_{ij}^{k_p}$ the special edges that h^t meets in order. For every $e_{ij}^{k_a}$, denote by $\widehat{P_{ij}^{k_a}}^t$ the special point on that edge (see definition in Section 7.1).

Claim (1): $d(h^t \cap e_{ij}^{k_0}, \widehat{P_{ij}^{k_0}}^t) = e^t d(h^0 \cap e_{ij}^{k_0}, \widehat{P_{ij}^{k_0}}^0)$. To see this, notice that by Lemma 5.9, $d(h^t \cap e_{ij}^{k_0}, \widehat{P_{ij}^{k_0}}^t)$ is equal to the parameter of the quadrilateral $\mathcal{Q}_i^t$, which is e^t times the parameter of the quadrilateral $\mathcal{Q}_i^0$, which again by Lemma 5.9 is equal to $d(h^0 \cap e_{ij}^{k_0}, \widehat{P_{ij}^{k_0}}^0)$.

Claim (2): $d(h^t \cap e_{ij}^{k_a}, \widehat{P_{ij}^{k_a}}^t) = e^t d(h^0 \cap e_{ij}^{k_a}, \widehat{P_{ij}^{k_a}}^0)$, for every special edge $e_{ij}^{k_a}$. By induction on a, this is true for the previous edge $e_{ij}^{k_{a-1}}$. For the edge $e_{ij}^{k_a}$ it then follows from Lemmas 5.11 or 5.9.

Claim (3): $d(h^t \cap b_{i+1}^j, O_{\mathcal{Q}_{i+1}^t}) = e^t d(h^0 \cap b_{i+1}^j, O_{\mathcal{Q}_{i+1}^0})$, which is the statement. We use Step (2) applied to the case $a = p$ (the last special edge), then we conclude as in Step (1), applying Lemma 5.9 to the quadrilaterals $\mathcal{Q}_{i+1}^t$ and $\mathcal{Q}_{i+1}^0$. ■

We are now ready to prove the stretch difference formula.

Proposition 7.12 (Stretch difference formula). The following holds:

$$\sum_{k=1}^{n_i} s^t(e_{ij}^k) - \sum_{k=1}^{n_i} e^t \cdot s^0(e_{ij}^k) = -\delta(e^t \cdot s_i) + e^t \cdot \delta(s_i) - \delta(e^t \cdot s_{i+1}) + e^t \cdot \delta(s_{i+1}) .$$

Proof. First compute $\sum_{k=1}^{n_i} s^t(e_{ij}^k)$ using (7):

$$\begin{aligned} \sum_{k=1}^{n_i} s^t(e_{ij}^k) &= \eta^t(O_{\mathcal{T}_i}) - O_{\mathcal{T}_{i+1}^t} = \\ &= [\eta^t(O_{\mathcal{T}_i^t}) - \eta^t(O_{\mathcal{Q}_i^t})] + [\eta^t(O_{\mathcal{Q}_i^t}) - O_{\mathcal{Q}_{i+1}^t}] + [O_{\mathcal{Q}_{i+1}^t} - O_{\mathcal{T}_{i+1}^t}] . \end{aligned} \tag{8}$$

Let us now compute separately each summand in the 2nd member:

$$\eta^t(O_{\mathcal{T}_i^t}) - \eta^t(O_{\mathcal{Q}_i^t}) = O_{\mathcal{Q}_i^t} - O_{\mathcal{T}_i^t} = -\delta(e^t \cdot s_i), \qquad \text{by Definition 7.9;}$$

$$\eta^t(O_{\mathcal{Q}_i^t}) - O_{\mathcal{Q}_{i+1}^t} = e^t \cdot [\eta^0(O_{\mathcal{Q}_i^0}) - O_{\mathcal{Q}_{i+1}^0}], \qquad \text{by Lemma 7.11;}$$

$$O_{\mathcal{Q}_{i+1}^t} - O_{\mathcal{T}_{i+1}^t} = -\delta(e^t \cdot s_{i+1}), \qquad \text{by Definition 7.9.}$$

Substituting the equations above in (8), we find

$$\sum_{k=1}^{n_i} s^t(e_{ij}^k) = -\delta(e^t \cdot s_i) - \delta(e^t \cdot s_{i+1}) + e^t \cdot [\eta^0(O_{\mathcal{Q}_i^0}) - O_{\mathcal{Q}_{i+1}^0}] . \tag{9}$$

Now we compute $\sum_{k=1}^{n_i} e^t \cdot s^0(e_{ij}^k)$ using (9) evaluated in $t=0$:

$$\sum_{k=1}^{n_i} e^t \cdot s^0(e_{ij}^k) = e^t \cdot \sum_{k=1}^{n_i} s^0(e_{ij}^k) = -e^t \cdot \delta(s_i) - e^t \cdot \delta(s_{i+1}) + e^t \cdot [\eta^0(O_{\mathcal{Q}_i}) - O_{\mathcal{Q}^0_{i+1}}] \,. \tag{10}$$

Combining (9) and (10), we get the following and we are done:

$$\sum_{k=1}^{n_i} s^t(e_{ij}^k) - \sum_{k=1}^{n_i} e^t \cdot s^0(e_{ij}^k) = -\delta(e^t \cdot s_i) + e^t \cdot \delta(s_i) - \delta(e^t \cdot s_{i+1}) + e^t \cdot \delta(s_{i+1}) \,. \qquad \blacksquare$$

7.4 Stretching the triangulated surface

In this section we will stretch the triangulated surface X_A using Bonahon's theory of cataclysms (see Section 3.4). For every $t \geq 0$ we will define a complete hyperbolic surface $((X_A)^t, \lambda_A, g^t)$ with a 1-1 local isometry $g^t : A^t \hookrightarrow (X_A)^t$, where A^t is the hyperbolic surface obtained stretching the auxiliary multi-cylinder along δ_A.

Definition 7.13 (Stretched auxiliary multi-cylinder). We define the *stretched auxiliary multi-cylinder* A^t and its maximal lamination δ_A by

$$A^t := \bigsqcup_j A_j^t \text{ and } \delta_A := \bigsqcup_j \delta_{A_j}.$$

It comes with an isometry $f^t := \bigsqcup_j f_j^t : C_j^t \to A_j^t$.

Notation 7.14. We will use the following notation:

- $\widehat{C_j^t} := f^t(C_j^t)$;
- $\widehat{C^t} := f^t(C^t)$;
- $\widehat{\Gamma^t} := f^t(\Gamma^t)$;
- $\widehat{R^t} := \overline{A^t \setminus \widehat{C^t}} = \overline{A^t \setminus f^t(C^t)}$.

7.4.1 *Construction of* $((X_A)^t, \lambda_A, g^t)$

By Proposition 7.4 we have that (X_A, λ_A) is a complete hyperbolic surface of finite volume with non-compact boundary and λ_A is a maximal lamination whose complement contains only triangles. We will first consider its double, that is, the surface (X_A^d, λ_A^d) defined as follows.

Definition 7.15 (Double of X_A). We define

$$X_A^d = X_A \sqcup X_A' / \sim,$$

where X_A' is an isometric copy of X_A with the opposite orientation, and $\sim$ identifies the boundary of X_A and X_A' with the identity map. The lamination λ_A^d is defined as

$$\lambda_A^d = \lambda_A \cup \lambda_A',$$

where λ_A' is the copy of λ_A on S_A'. If $A_1, \ldots, A_m$ are the auxiliary cylinders in X_A, then $A_{m+1}, \ldots, A_{2m}$ are the auxiliary cylinders in X_A' (A_{i+m} is the mirror copy of A_i).

The following fact is immediate.

Proposition 7.16. The surface X_A^d is a finite hyperbolic surface without boundary and λ_A^d is a maximal lamination on X_A^d.

For every $t \geq 0$ we will now define a new hyperbolic structure $(X_A^d)^t$ by defining a suitable cocycle for λ_A^d (see Bonahon's Theorem 3.7). Let ρ^0 be the shearing cocycle for the lamination λ_A^d associated to the hyperbolic structure X_A^d. Note that for every $t \geq 0$ the cocycle $e^t \cdot \rho^0$ is the cocycle of the hyperbolic structure $(X_A^d)^t_{Th}$ obtained via the Thurston stretch of X_A^d. We will define our hyperbolic structure $(X_A^d)^t$ on X_A^d by adding a term to the cocycle $e^t \cdot \rho^0$.

Definition 7.17 (Cocycle ρ^t on λ_A^d). Choose a train track τ snugly carrying λ_A^d such that τ contains one subtrack τ_{ij} as in Fig. 18 for every spike α_i^j in one of the A_j, for $j \in \{1, \ldots, 2m\}$. (Here we label every edge of τ_{ij} by the unique edge of $\delta_{A_j} \cup \partial A_j$ it carries, and the switch v_i^j corresponds to the spike α_i^j.)

We define an assignment of real weights ϵ^t on the edges of τ. Define $\epsilon^t(e) := 0$ for every $e \in \tau$ such that $e \notin \bigcup_{ij} \tau_{ij}$. For $e \in \tau_{ij}$ the assignment $\epsilon^t(e)$ is the following:

$$\epsilon^t(a_i^j) := 0, \tag{11}$$

$$\epsilon^t(e_{ij}^k) := -e^t s^0(e_{ij}^k) + s^t(e_{ij}^k) \text{ for } k = 1, \ldots, n_i, \tag{12}$$

$$\epsilon^t(b_i^j) := \delta(e^t \cdot s_i) + e^t \cdot \delta(s_i), \tag{13}$$

$$\epsilon^t(b_{i+1}^j) := \delta(e^t \cdot s_{i+1}) + e^t \cdot \delta(s_{i+1}), \tag{14}$$

where the functions s^t and δ were defined in Section 7.3. Now we define $(\rho^t(e))_{e\in\tau}$ as

$$\rho^t(e) := e^t \cdot \rho^0(e) + \epsilon^t(e)\,.$$

In the next section we will prove the following:

Proposition 7.18. For every $t \geq 0$, the assignment of real weights ρ^t on the edges of τ defines the shearing cocycle for the lamination λ_A^d of a hyperbolic structure on X_A^d, which we will denote by $(X_A^d)^t$.

The lamination λ_A^d and the cocycle ρ^t are both symmetric for the involution of X_A^d, hence the hyperbolic structure $(X_A^d)^t$ is also symmetric.

Definition 7.19 (Triangulated stretched surface). The *triangulated stretched surface* $(X_A)^t$ is the restriction of the hyperbolic structure $(X_A^d)^t$ to the surface X_A. (Note that X_A^t is a complete hyperbolic surface of finite volume.)

Proposition 7.20. There is a 1-1 local isometry $g^t : A^t \hookrightarrow (X_A)^t$. When $t = 0$ we have $((X_A)^0, \lambda_A, g^0) = (X_A, \lambda_A, g)$.

Proof. By definition, for every $e_{ij}^k \in \delta_A$ we have

$$\rho^t(e_{ij}^k) = e^t \cdot \rho^0(e_{ij}^k) + \epsilon^t(e_{ij}^k) = e^t \cdot s^0(e_{ij}^k) + (-e^t s^0(e_{ij}^k) + s^t(e_{ij}^k)) = s^t(e_{ij}^k)\,,$$

which are the shear coordinates of A^t for the ideal triangulation δ_A. ■

We also want to construct a stretch map for the stretched triangulated surface $(X_A)^t$. This will be given by the composition of Thurston's stretch map and a shear map. Consider the two hyperbolic surfaces $(X_A^d)^t_{Th}$ and $(X_A^d)^t$. We will denote by κ^t the *shear map* between them with respect to the lamination λ_A^d:

$$\kappa^t : (X_A^d)^t_{Th} \setminus \lambda_A^d \to (X_A^d)^t \setminus \lambda_A^d.$$

Every triangle in the complement of λ_A^d in $(X_A^d)^t_{Th}$ is mapped isometrically to the corresponding triangle in the complement of λ_A^d in $(X_A^d)^t$ (see Bonahon [2, Sec. 4]). Thurston [20] called this map a *cataclysm*.

Consider the sublamination $\mu_A \subset \lambda_A$ defined in Definition 7.2. This is the closure of the image of the lamination δ_A. Its double, μ_A^d is a lamination on X_A^d.

Lemma 7.21. The map κ^t extends continuously to an isometry

$$\bar{\kappa}^t : (X_A^d)^t_{Th} \setminus \mu_A^d \to (X_A^d)^t \setminus \mu_A^d \ .$$

Proof. Notice that $(X_A^d)^t_{Th} \setminus \mu_A^d$ is an open subset of $(X_A^d)^t_{Th}$. Given a point $x \in (X_A^d)^t_{Th} \setminus \mu_A^d$, we can choose a ball B centered at x and contained in $(X_A^d)^t_{Th} \setminus \mu_A^d$. Notice that the cocycles associated to the two hyperbolic structures $(X_A^d)^t_{Th}$ and $(X_A^d)^t$ differ by the cocycle ϵ^t, which is supported in μ_A. Hence, when we restrict our attention to the ball B, the two cocycles agree. To prove the proposition in the ball B, we can then proceed as in the proof of [2, Lemma 11]. The proof given there uses the horocyclic foliation of a triangulated surface, see also Section 8.2.2 where we will generalize that notion for surfaces with boundary. ∎

We are now ready to construct the stretch map for $(X_A)^t \setminus \mu_A$.

Proposition 7.22 (Existence of a stretch map for $X_A \setminus \mu_A$). For every $t \geq 0$, there exists a continuous map $\psi^t : X_A \setminus \mu_A \to (X_A)^t \setminus \mu_A$ homotopic to the identity with the following properties:

1. ψ^t stretches the arc length of the leaves of $\lambda_A \setminus \mu_A$ by e^t;
2. on every triangular geometric piece $\mathcal{T}$ in $X_A \setminus \lambda_A$, the map ψ^t restricts to $\psi^t|_{\mathcal{T}} = \phi^t : \mathcal{T} \to \mathcal{T}^t$ as in Lemma 5.5;
3. ψ^t is locally Lipschitz with local Lipschitz constant equal to e^t.

Proof. We will denote Thurston's stretch map by

$$\tau^t : X_A^d \to (X_A^d)^t_{Th}.$$

This map was introduced in [20]. On every triangle in $X_A^d \setminus \lambda_A^d$, this map agrees with the map ϕ^t from Lemma 5.5. The map τ^t is continuous, stretches the arc length of the leaves of λ_A by e^t and $\mathrm{Lip}(\tau^t) = e^t$. We define the map ψ^t as follows:

$$\psi^t = \bar{\kappa}^t \circ \tau^t : X_A \setminus \mu_A \to (X_A)^t \setminus \mu_A.$$

It satisfies the stated properties because of the properties of τ^t and Lemma 7.21. ∎

7.4.2 *Proof of Proposition 7.14*

Lemma 7.23. The assignments of real weights $(\epsilon^t(e))_{e\in\tau}$ on the edges of τ defines a transverse cocycle for the lamination λ_A^d.

Proof. By Theorem 3.7 we need to check that the switch relations hold at every switch $v \in \tau$. First, assume $v = v_i^j \in \tau_{ij}$ as in Fig. 18. We will check the switch relation:

$$\epsilon^t(a_i^j) = \epsilon^t(b_i^j) + \epsilon^t(b_{i+1}^j) + \sum_{k=1}^{n_i} \epsilon^t(e_{ij}^k) \ . \tag{15}$$

This equation is satisfied because it is equivalent to Lemma 7.12:

$$\epsilon^t(b_i^j) + \epsilon^t(b_{i+1}^j) + \sum_{k=1}^{n_i} \epsilon^t(e_{ij}^k) = 0 \ . \tag{16}$$

If v is a switch of τ but not a switch of τ_{ij} then $\epsilon^t(e) = 0$ for every edge e of τ concurring in the switch. Therefore, the switch condition at v is satisfied, and we conclude. ■

Lemma 7.24. The assignments of real weights $(\rho^t(e))_{e\in\tau}$ on the edges of τ defines a transverse cocycle for the lamination λ_A^d.

Proof. By Lemma 7.24, we have $\epsilon^t \in H(\lambda_A^d, \mathbb{R})$, so ρ^t is a linear combination of transverse cocycles for λ_A^d. As $H(\lambda_A^d, \mathbb{R})$ is a vector space by Theorem 3.7, we have $\rho^t \in H(\lambda_A^d, \mathbb{R})$. ■

Lemma 7.25. For every measure μ on λ_A^d, we have $\omega(\epsilon^t, \mu) = 0$.

Proof. We compute $\omega(\epsilon^t, \mu)$ using Lemma 3.9. After a finite sequence of splittings, τ can be made generic. In particular, splitting each subtrack τ_{ij} we get to a generic subtrack τ'_{ij} as in Fig. 22. Note that if v is a switch and $v \notin \tau'_{ij}$ then $\epsilon^t(e_v^r) = \epsilon^t(e_v^l) = 0$ because $\epsilon^t(e) = 0$ for every edge $e \notin \bigcup \tau_{ij}$. Therefore, we just need to look at all the switches $v \in \tau'_{ij}$, and we have

$$\omega(\epsilon^t, \mu) = \sum_j \sum_i \sum_{w\in\tau'_{ij}} [\epsilon^t(e_w^r)\mu(e_w^l) - \epsilon^t(e_w^l)\mu(e_w^r)].$$

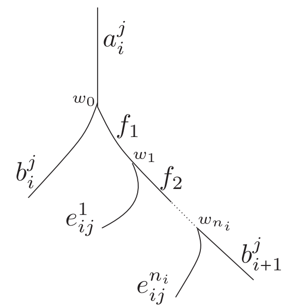


Fig. 22. Splitting τ_{ij} to make it generic.

Note that for every measure μ on λ_A^d, we must have $\mu(e_{ij}^k) = 0$ for all $k = 1, \ldots, n_i$. Indeed, all of the e_{ij}^k's enter the same spike $a_i^j \in A_j \subset X_A$, and either all the e_{ij}^k's enters the same cusp or they accumulate on the same sublamination of λ_A^d. In either case for each of them $\mu(e_{ij}^k) = 0$. It follows that at every switch $w \neq w_0 \in \tau'_{ij}$ we have

$$\epsilon^t(e_w^r)\mu(e_w^l) = 0. \tag{17}$$

Moreover, by the switch relations for ϵ^t, we have

$$\epsilon^t(f_1) = \epsilon^t(a_i^j) - \epsilon^t(b_i^j) = -\epsilon^t(b_i^j) \tag{18}$$

$$\epsilon(b_i^j) + \epsilon^t(e_{ij}^1) + \ldots + \epsilon^t(e_{ij}^{n_i}) = \epsilon^t(a_i^j) - \epsilon^t(b_{i+1}^j) = -\epsilon^t(b_{i+1}^j) \tag{19}$$

Using (17), (18), and (19) in the computation of $\omega(\epsilon^t, \mu)$, we have

$$\begin{aligned}
\omega(\epsilon^t, \mu) &= \sum_j \sum_i \sum_{w \in \tau'_{ij}} [\mu(e_w^l)\epsilon^t(e_w^r) - \epsilon^t(e_w^l)\mu(e_w^r)] \\
&= \sum_j \sum_i \Big[\mu(b_i^j)\epsilon^t(f_1) - \mu(b_{i+1}^j)[\epsilon^t(b_i^j) + \epsilon^t(e_{ij}^1) + \ldots + \epsilon^t(e_{ij}^{n_i})]\Big] && \text{by (17)} \\
&= \sum_j \sum_i \Big[-\mu(b_i^j)\epsilon^t(b_i^j) + \mu(b_{i+1}^j)\epsilon^t(b_{i+1}^j)\Big] && \text{by (18) and (19)} \\
&= \sum_j 0 = 0\,.
\end{aligned}$$

Indeed, for every j we have

$$\sum_i \left[-\mu(b_i^j)\epsilon^t(b_i^j) + \mu(b_{i+1}^j)\epsilon^t(b_{i+1}^j) \right] = 0$$

because the b_i^j's form the cycle $c_j = \bigcup_i b_i^j \subset \partial A_j^\Sigma$. ■

Proof of Proposition 7.14. Note that ρ^0 is the shearing cocycle associated with a hyperbolic structure, hence by Theorem 3.10 it satisfies

$$\omega(\rho^0, \alpha) > 0$$

for every transverse measure α on λ_A^d. By the bi-linearity of ω and Lemma 7.25, for every transverse measure α on λ_A^d we have

$$\omega(e^t \cdot \rho^0 + \epsilon^t, \alpha) = e^t \cdot \omega(\rho^0, \alpha) + \omega(\epsilon^t, \alpha) = e^t \cdot \omega(\rho^0, \alpha) > 0.$$

The statement then follows from Theorem 3.10. ■

8 Generalized Stretch Lines

In this section we finally prove Theorem 1.1.

8.1 Generalized stretch lines

Let $X \in \text{Teich}(S)$ and λ a maximal lamination on X. In this section we will define the generalized stretch line starting from X and directed by λ: for every $t \geq 0$ we will define an element $X_\lambda^t \in \text{Teich}(S)$.

In Section 7.4, we defined the stretched triangulated surface $(X_A)^t$, with a 1-1 local isometry of the stretched auxiliary multicylinder:

$$g_t : A^t \to (X_A)^t.$$

Recall that

$$A^t = \widehat{C}^t \cup \widehat{R}^t,$$

where $\widehat{C}^t$ is the union of m cylinders $\widehat{C}_j^t$ that are isometric to cylinders in the stretched boundary block B^t (see Fig. 15.)

We define

$$(X_C)^t := \overline{(X_A)^t \setminus g^t(\widehat{R^t})} \subset (X_A)^t.$$

This is a hyperbolic structure on a surface with boundary homeomorphic to X_C. It contains a copy of C^t, we will denote the 1-1 local isometry by $h^t := g^t \circ f^t : C^t \hookrightarrow (X_C)^t$.

We can now define the hyperbolic structure X^t_λ, for every $t \geq 0$:

$$X^t_\lambda := B^t \sqcup (X_C)^t / \sim,$$

where $\sim$ identifies a point $z \in C^t$ with the point $h^t(z) \in (X_C)^t$.

Proposition 8.1. Let $\pi : B^t \cup (X_C)^t \to X^t_\lambda$ be the projection map. The following diagram is commutative with all arrows 1-1 local isometries, therefore X^t_λ is a hyperbolic structure on S:

$$\begin{tikzcd} & B^t \arrow[dr, "\pi|"] & \\ C^t \arrow[ur, "\iota"] \arrow[dr, "h^t"'] & & X^t_\lambda \\ & (X_C)^t \arrow[ur, "\pi|"'] & \end{tikzcd}$$

Notice that the lamination λ on X^t_λ is the union

$$\lambda = \pi(\lambda_B) \cup \pi(\lambda_{X_C}).$$

Consider the set $\pi(\partial^{nc} B^t)$, the image of the union of the non-compact boundary components of B^t. This set is a union of finitely many geodesics, but it is in general not closed. Its closure is a sublamination of λ that we will call μ_X:

$$\mu_X = \overline{\pi(\partial^{nc} B^t)}.$$

We also denote by ν_X the lamination

$$\nu_X = \mu_X \setminus \pi(\partial^{nc} B^t).$$

Notice that

$$\nu_X \subset \mu_X \subset \lambda.$$

These sublaminations are closely related to μ_A, ν_A from Definition 7.3.

Definition 8.2. For every $X \in \text{Teich}(S)$ and λ maximal lamination, the line

$$s_{X,\lambda} : \mathbb{R}_{\geq 0} \ni t \to X^t_\lambda \in \text{Teich}(S)$$

is a *generalized stretch line* starting from X and directed by λ.

8.2 Generalized stretch maps

We will now see that the generalized stretch lines verify Theorem 1.1. We will need to define the *generalized stretch map* $\Phi^t : X \to X^t_\lambda$.

8.2.1 *The generalized stretch map on an open dense subset*

We will first define a map in an open dense subset, and later we will extend it everywhere. Consider the map

$$\alpha^t : X \setminus \nu_X \to X^t_\lambda \setminus \nu_X$$

defined as follows:

$$\alpha^t(z) := \begin{cases} \beta^t(z) & \text{if } z \in B, \\ \psi^t(z) & \text{if } z \notin B, \end{cases} \tag{20}$$

where $\beta^t : B \to B^t$ is the map constructed in Proposition 6.7, and $\psi^t : X_A \setminus \mu_A \to (X_A)^t \setminus \mu_A$ is the map given by Proposition 7.22.

Lemma 8.3. The map α^t is well defined on $X \setminus \nu_X$ and continuous.

Proof. Notice that ψ^t is defined in $X_A \setminus \mu_A$. Every point of X coming from μ_A is either in B or in ν_X, hence α^t is well defined on $X \setminus \nu_X$. We need to check the continuity of the map at the points of $\partial^{nc}B$. Every connected component b of $\partial^{nc}B$ is the bi-infinite edge of a quadrilateral piece $\mathcal{Q}$, with shear parameter s. The geodesic b is part of the lamination λ_A of X_A, hence the map ψ^t is not defined on b. We can extend it to b in two ways: we denote by $\underline{\psi}^t$ the extension from B, and by $\overline{\psi^t}$ the extension from $X \setminus B$. Both extensions are mapping b to b^t, the copy of b in X^t_λ:

$$\underline{\psi}^t, \overline{\psi^t} : b \to b^t$$

$$\underline{\psi}^t(z) := \lim_{m \to +\infty} \psi^t(w_m), \qquad \text{where } \{w_m\}_{m \in \mathbb{N}} \subset \mathcal{Q} \text{ such that } w_m \to z \text{ as } m \to +\infty$$

$$\overline{\psi^t}(z) := \lim_{m \to +\infty} \psi^t(w_m), \qquad \text{where } \{w_m\}_{m \in \mathbb{N}} \subset X \setminus B \text{ such that } w_m \to z \text{ as } m \to +\infty$$

By the definition of ψ^t using shear maps and by the definition of the cocycle ϵ^t, we have

$$\underline{\psi}^t(z) - \overline{\psi^t}(z) = \delta(e^t \cdot s) - e^t \delta(s)\,, \tag{21}$$

using Notation 7.8. Now we claim that, for every $z \in b$, we have

$$\beta^t(z) - \underline{\psi}^t(z) = -\delta(e^t \cdot s) + e^t \delta(s)\,. \tag{22}$$

In order to see this, notice that

1. the map $\underline{\psi}^t : b \to b^t$ fixes $O_{\mathcal{T}^0}$ and stretches the arc length of b by a factor e^t:

$$\forall P \in b_i \quad : \quad [\underline{\psi}^t(P) - O_{\mathcal{T}^0}] = e^t \cdot (P - O_{\mathcal{T}^0}) \quad \text{with } \underline{\psi}^t(O_{\mathcal{T}^0}) = O_{\mathcal{T}^t}$$

2. the map $\beta^t : b \to b^t$ fixes $O_{\mathcal{Q}}$ and stretches the arc length of b by a factor e^t:

$$\forall P \in b \quad : \quad [\beta^t(P) - O_{\mathcal{Q}^t}] = e^t \cdot (P - O_{\mathcal{Q}^0}) \quad \text{with } \beta^t(O_{\mathcal{Q}^0}) = O_{\mathcal{Q}^t}$$

Putting these formulas together we have

$$[\beta^t(P) - \underline{\psi}^t(P)] = [O_{\mathcal{Q}^t} - O_{\mathcal{T}^t}] + e^t \cdot [O_{\mathcal{Q}^0} - O_{\mathcal{T}^0}] = -\delta(e^t \cdot s) + e^t \cdot \delta(s)\,,$$

which proves the claim. Now (21) and (22) imply that $\beta^t(z) - \overline{\psi^t}(z) = 0$, hence $\beta^t(z) = \overline{\psi^t}(z)$. This shows the continuity of the map α^t on $X \setminus \nu_X$. ∎

8.2.2 *The horocyclic foliation on X*

We will now define a partial foliation on X, called the horocyclic foliation and denoted by $\mathcal{K}$. Denote by $\{\mathcal{G}_i\}$ the finite set of geometric pieces of $X \setminus \lambda$. For every $\mathcal{G}_i$, we defined a horocyclic foliation $\mathcal{K}_i$ in Definition 5.6, 5.15, and 5.17. If $\mathcal{G}_i$ is an hexagonal piece, $\mathcal{K}_i$ is empty. For every $\mathcal{G}_i$, we denote by K_i the support of $\mathcal{K}_i$. The support of $\mathcal{K}$ will be the set

$$K = \overline{\bigcup_i K_i}.$$

On $K \cap B$, we define the partial foliation by glueing the partial foliations $\mathcal{K}_i$ on the pieces $\mathcal{G}_i$ that are in B. To define the partial foliation on $K \setminus B$, we notice that $K \setminus B \subset X_A$. The double $(X_A)^d$ is a finite hyperbolic surface without boundary, hence we can apply Thurston's theory [20], and consider Thurston's horocyclic foliation on $(X_A)^d$. The set $K \setminus B$ is contained in the support of Thurston's foliation, and we define our foliation on $K \setminus B$ as the restriction of Thurston's foliation. This defines a partial foliation on X whose support is K. We will call it the *horocyclic foliation* on X, and denote it by $\mathcal{K}$. By definition, for every $\mathcal{G}_i$, the restriction of $\mathcal{K}$ to $\mathcal{G}_i$ coincides with $\mathcal{K}_i$.

We will now describe how $\mathcal{K}$ looks like locally in the neighborhood of every point. If a point lies in $X \setminus \lambda$, then it is in the interior of a piece $\mathcal{G}_i$. In this case we know that the horocyclic foliation around this point looks like one of the explicit models given in Definition 5.6, 5.15, or 5.17. If the point is on λ, then it lies on a geodesic $\ell \subset \lambda$. For every side of ℓ, there can be a geometric piece bounded by ℓ on that side or not. If there is a geometric piece, then again $\mathcal{K}$ looks like one of the explicit models on that side of ℓ. If there is no geometric piece on that side of ℓ, the situation is even simpler, as we now describe. For $z \in \ell$, a small ball centered at z is divided by ℓ in two parts, which we call *half-balls*, one on every side of ℓ.

Lemma 8.4. Let $z \in \ell \subset \lambda$. If on one side ℓ does not bound a geometric piece, then there exists a small half-ball U centered at z on that side of ℓ such that U is completely foliated by $\mathcal{K}$ with leaves that hit orthogonally ℓ and all the leaves of $U \cap \lambda$.

Proof. Assume that the radius of U is small so that U does not intersect ∂X nor any leaf of λ of finite length. In particular, U does not intersect any hexagonal piece.

Now, let us work in the universal covering $\widetilde{X} \subset \mathbb{H}^2$. We denote by $\widetilde{U}$ a lift of U. Every connected component of $\widetilde{X} \setminus \widetilde{\lambda}$ is a geometric piece that is a copy of one of the $\mathcal{G}_i$.

If we assume that the radius of U is also smaller than $\frac{1}{2}\log(3)$ (the radius of the inscribed circle to an ideal triangle), then $\widetilde{U}$ intersects at most two edges of every geometric piece, these two edges meet at an ideal vertex of the piece. There is at most one piece, say $\widetilde{\mathcal{G}}_0$, such that $\widetilde{U}$ intersects only one edge of $\widetilde{\mathcal{G}}_0$, see Fig. 23.

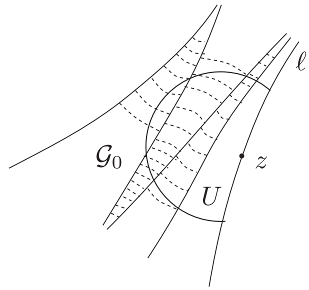


Fig. 23. Lemma 8.4.

There are only finitely many $\mathcal{G}_i$, with finitely many values of the parameters. The lengths of the horocycles of the $\mathcal{K}_i$ passing through the points O_T^j for a triangular piece, O_C, O_D for a quadrangular piece, O_W for a pentagonal piece have a minimal value D.

We can also assume that the radius of U is smaller than $D/2$, this is the situation represented in Fig. 23. Now if $\widetilde{U}$ meets two edges of a geometric piece, then the intersection of $\widetilde{U}$ with the piece is completely contained in the support of the horocyclic foliation. This might still be false for $\widetilde{\mathcal{G}}_0$, but up to reducing again the radius of U, we can make sure that $\widetilde{U}$ does not intersect this piece at all. With this choice of the radius, we have that $U \setminus \lambda$ is completely foliated by $\mathcal{K}$, with a foliation made by pieces of horocycles. This foliation extends nicely to all the leaves of λ contained in U and it is perpendicular to them, this is proved in [2, Section 2]. ∎

The definition of the horocyclic foliation works for every finite hyperbolic surface equipped with a maximal lamination, in particular for X_λ^t, $(X_A^d)^t$, and $(X_A^d)_{Th}^t$.

Lemma 8.5. Let $z \in \ell$, for a geodesic $\ell \subset \nu_X$. Let U be a half-ball centered at z as in Lemma 8.4. If $x, y \in U \setminus \nu_X$ are in the same leaf for the restriction of $\mathcal{K}$ to U, then their images $\alpha^t(x), \alpha^t(y)$ are in the same leaf for the horocyclic foliation on X_λ^t.

Proof. If U satisfies the thesis of Lemma 8.4, it is contained in X_C. Let $f \subset U$ be the leaf for the restriction of $\mathcal{K}$ to U that contains x, y. Every component of $f \setminus \lambda$ is contained in a geometric piece, hence it is mapped by α^t to a leaf of $\mathcal{K}$.

Let us first assume that x, y are not in B. Recall that α^t on them is defined as $\psi^t = \bar{\kappa}^t \circ \tau^t$. Consider the arc $f_{x,y}$ of f between x and y. When $f_{x,y}$ is considered as a subset of X_A^d, it lies in a leaf of the horocyclic foliation of X_A^d. Thurston's stretch map $\tau^t : X_A^d \to (X_A^d)_{Th}^t$ sends $f_{x,y}$ in a leaf of the horocyclic foliation of $(X_A^d)_{Th}^t$, hence $\tau^t(x)$ and

$\tau^t(y)$ are in the same leaf in $(X_A^d)^t_{Th}$. We have to check that $\bar{\kappa}^t$ also sends $\tau^t(x)$ and $\tau^t(y)$ to the same leaf in $(X_A^d)^t$. In order to do this, notice that the horocyclic distance between $\bar{\kappa}^t(\tau^t(x))$ and $\bar{\kappa}^t(\tau^t(y))$ is equal to the measure of the arc $\tau^t(f_{x,y})$ for ϵ^t (see the definition of cocycle associated to a hyperbolic metric in [2, Section 2]). Since x, y do not lie in B, the measure of this arc for ϵ^t is zero by definition of ϵ^t.

Let us now prove the statement when x, y are in B. Notice that every component of $f \setminus \lambda$ is mapped by α^t to a leaf of $\mathcal{K}$. By the continuity of α^t on $X \setminus \nu_X$ (Lemma 8.3), every connected component of $f \setminus \nu_X$ is mapped by α^t to a leaf of $\mathcal{K}$. ■

Proposition 8.6. The map $\alpha^t : X \setminus \nu_X \to X^t_\lambda \setminus \nu_X$ extends to a continuous map

$$\Phi^t : X \to X^t_\lambda \ .$$

Proof. For every point $z \in \nu_X$, consider a small half-ball U centered in z as in Lemma 8.5. If f is the leaf of $\mathcal{K}$ through z, then $\alpha^t(f)$ lies on a leaf of the horocyclic foliation on X^t_λ. We define $\Phi^t(z)$ as the point of $\nu_X \subset X^t_\lambda$ lying on the leaf containing $\alpha^t(f)$. The map Φ^t maps U homeomorphically to a half-neighborhood of $\Phi^t(z)$ in X^t_λ. ■

We are ready to prove our main theorem:

Theorem 1.1. Let S be a surface with non-empty boundary and fix $X \in \mathrm{Teich}(S)$. For every maximal lamination λ on X and for every $t \geq 0$ there exists $X^t_\lambda \in \mathrm{Teich}(S)$ and a Lipschitz map $\Phi^t : X \to X^t_\lambda$, called *generalized stretch map*, with the following properties:

1. $X^0_\lambda = X$;
2. $\mathrm{Lip}\,(\Phi^t) = e^t$;
3. $\Phi^t(\partial X) = \partial X^t_\lambda$;
4. Φ^t stretches the arc length of the leaves of λ by the factor e^t;
5. for every geometric piece $\mathcal{G}$ in $X \setminus \lambda$, the map Φ^t restricts to a generalized stretch map $\phi^t : \mathcal{G} \to \mathcal{G}_t$ as described in Lemmas 5.5 and 5.8;
6. if λ contains a non-empty measurable sublamination, we have

$$\mathrm{Lip}(\Phi^t) = \min\,\{\mathrm{Lip}(\psi) \mid \psi \in \mathrm{Lip}_0(X, X^t_\lambda), \psi(\partial X) \subset \partial X^t_\lambda\},$$

where $\mathrm{Lip}_0(X, Y)$ is the set of all Lipschitz maps homotopic to the identity.

Proof. Properties (1)-(5) follow from the construction of Φ^t. For Property (6), from Property (2) we have

$$d_A(X, X_\lambda^t) \leq d_{L\partial}(X, X_\lambda^t) \leq t\,.$$

The arc length on λ is multiplied by e^t by Property (4), and since λ contains a measurable sublamination we have, by Theorem 3.4, that $d_A(X, X_\lambda^t) \geq t$. This implies

$$d_A(X, X_\lambda^t) = d_{L\partial}(X, X_\lambda^t) = t\,. \qquad \blacksquare$$

9 The Geometry of the Arc Distance

In this section we will prove the corollaries of Theorem 1.1 stated in the introduction.

9.1 Stretch lines are geodesics

We will now prove that, if a lamination λ contains a measurable sublamination, then a generalized stretch line is a geodesic in $\mathrm{Teich}(S)$ for both the arc distance d_A and the Lipschitz distance $d_{L\partial}$.

Corollary 1.2. For every $X \in \mathrm{Teich}(S)$ and every maximal lamination λ on X, if λ contains a non-empty measurable sublamination then the *generalized stretch line*

$$\begin{aligned} s_{X,\lambda} : \mathbb{R}_{\geq 0} &\longrightarrow \mathrm{Teich}(S) \\ t &\mapsto X_\lambda^t \end{aligned}$$

is a geodesic path parametrized by arc length for both d_A and $d_{L\partial}$.

Proof. This follows from Theorem 1.1, once we notice that $\left(X_\lambda^{t_1}\right)_\lambda^{t_2 - t_1} = X_\lambda^{t_2}$. $\blacksquare$

9.2 The Teichmüller space is geodesic

We will now prove that every pair of points $X, Y \in \mathrm{Teich}(S)$ is connected by a path that is geodesic for both distances d_A and $d_{L\partial}$. The path will be a finite concatenation of generalized stretch segments. Notice that, in a Riemannian manifold, a concatenation of geodesic segments coming from distinct geodesic cannot be a geodesic. These distances are indeed not induced by a Riemannian metric, we will see later that they are instead induced by a Finsler metric. In a Finsler manifold, a geodesic segment might admit several geodesic extensions.

The 1st ingredient for the proof is the notion of ratio-maximizing measured lamination. We will first recall Thurston's definition for closed or punctured surfaces and we then extend these notions to surfaces with boundary.

Definition 9.2 (Ratio-maximizing lamination for closed or punctured surfaces [20]). Let S be a closed or punctured surface. Fix $X, Y \in \mathrm{Teich}(S)$. A geodesic lamination μ is a *ratio-maximizing* for X, Y if there exists a homeomorphism f from a neighborhood of μ in X to a neighborhood of μ in Y such that

1. f is R-Lipschitz, where $R := \exp(d_A(X, Y))$;
2. f is homotopic to the identity;
3. f maps the support of μ in X to the support of μ in Y stretching the arc length of μ affinely by a factor R.

Thurston [20] proves that for every pair of points $X, Y \in \mathrm{Teich}(S)$ there exists a unique *largest ratio-maximizing lamination* $\mu(X, Y)$.

Definition 9.3 (Ratio-maximizing lamination for surfaces with boundary). Let S be a surface with boundary and fix $X, Y \in \mathrm{Teich}(S)$. A geodesic lamination μ is a *ratio-maximizing lamination* for X and Y if μ^d is ratio-maximizing for X^d, Y^d. Moreover, consider the unique largest ratio-maximizing lamination $\mu(X^d, Y^d)$ in S^d. By uniqueness, $\mu(X^d, Y^d)$ is symmetric and restricts to a lamination on S that we denote by $\mu(X, Y)$ and call the *largest ratio-maximizing lamination* for $X, Y \in \mathrm{Teich}(S)$.

Proposition 9.4. Let $X, Y \in \mathrm{Teich}(S)$ and let μ be a measured lamination. Then, the support of μ is ratio-maximizing for X, Y if and only if μ realizes the maximum in the formula for $d_A(X, Y)$ given in Theorem 3.4.

Proof. If S is closed or punctured, $d_A = d_{Th}$ and the result was proven by Thurston [20]. If S has boundary we can conclude by a doubling argument: the support of μ^d is ratio-maximizing for X^d, Y^d hence it realizes the maximum for d_{Th} and by Proposition 2.10, μ realizes the maximum for d_A. ■

Lemma 9.5. Let λ be a maximal lamination, and $X \in \mathrm{Teich}(S)$. Then for all $t \geq 0$,

$$\mu(X, X^t_\lambda) = \lambda, .$$

Proof. We need to construct a suitable homeomorphism ϕ from a neighborhood M_X of λ in X to a neighborhood $M_{X^t_\lambda}$ of λ in X^t_λ. We initially describe ϕ in every geometric piece. For the triangular pieces, ϕ will agree with Thurston's stretch map ϕ^t. For the other pieces, ϕ agrees with our stretch maps ϕ^t on the support of the horocyclic foliation $\mathcal{K}$ and on the edges that are leaves of λ. We can always extend it to a small neighborhood of these edges by a homeomorphism. In the rest of X, ϕ is defined only on the support of $\mathcal{K}$, and there it agrees with our stretch map Φ^t. We know that Φ^t is a homeomorphism on the support of $\mathcal{K}$ (proof of Proposition 8.6). Using Lemma 8.4, we see that the union of the support of $\mathcal{K}$ with the triangular pieces and with a small neighborhood of the edges of the geometric pieces part of λ is a neighborhood of λ. ■

Lemma 9.6. The lamination $\mu(X, Y)$ contains a measurable sublamination.

Proof. This is because the maximum in the formula for $d_A(X, Y)$ given in Theorem 3.4 is always achieved by some measurable lamination, whose support is ratio-maximizing by Proposition 9.4, hence contained in $\mu(X, Y)$. ■

The following lemma is a simple adaptation of a result of Thurston.

Lemma 9.7. Let $X, Y \in \mathrm{Teich}(S)$. If X_i and Y_i are sequences of hyperbolic structures converging to X and Y, then $\mu(X, Y)$ contains every lamination in the limit set of $\mu(X_i, Y_i)$ in the Hausdorff topology.

Proof. If S has no boundary, see Thurston [20, Theorem 8.4]. If S has boundary, it follows from Thurston's result via a doubling argument. ■

Given $X, Y \in \mathrm{Teich}(S)$, we will now construct a geodesic segment joining them. Our proof goes along similar lines as the proof of Thurston's [20, Theorem 8.5]. In our case, we need to be more careful because our generalized stretch maps are not known to be homeomorphisms everywhere, in contrast to Thurston's stretch maps. Our maps are known to be homeomorphisms only when restricted to the subset K from Section 8.2.2, we already used this fact in Lemma 9.5.

The idea of the proof is the following: if $\mu(X, Y)$ is a maximal lamination, we can simply consider the generalized stretch line starting at X with respect to $\mu(X, Y)$, and prove that it passes through Y. If $\mu(X, Y)$ is not maximal, we will first complete it to a maximal lamination $\lambda \supset \mu(X, Y)$, and consider the generalized stretch line starting at X

with respect to λ. This will usually not pass through Y, hence we need to stop following this geodesic at some point, and start following another one.

Lemma 9.8. Let λ be a maximal lamination containing $\mu(X, Y)$. There exists ϵ such that for every $0 < t < \epsilon$ we have

1. $\mu(X^t_\lambda, Y) = \mu(X, Y)$;
2. $d_A(X^t_\lambda, Y) = d_A(X, Y) - t$.

Proof. Let μ denote $\mu(X, Y)$. By definition of μ, there exist neighborhoods N_X, N_Y of μ in X and Y respectively and a Lipschitz homeomorphism $f : N_X \to N_Y$ with $\mathrm{Lip}(f) = e^{d_A(X,Y)}$ mapping μ to itself and stretching its arc length affinely by $e^{d_A(X,Y)}$.

By Lemma 9.5, there exists two neighborhoods $M_X, M_{X^t_\lambda}$ of μ in X, X^t_λ respectively and a homeomorphism $\phi : M_X \to M_{X^t_\lambda}$ with $\mathrm{Lip}(\phi) = e^t$ and ϕ maps μ to itself by affinely stretching it by e^t. The composition

$$f' = f \circ \phi^{-1} : M_{X^t_\lambda} \to N_Y$$

has $\mathrm{Lip}(f') = e^{d_A(X,Y)-t}$ and maps μ to itself affinely stretching by $e^{d_A(X,Y)-t}$.

By Lemma 9.7 there exists ϵ such that if $0 < t < \epsilon$ then $\mu(X^t_\lambda, Y) \subset N_X$. Since $\mathrm{Lip}(f') = e^{d_A(X,Y)-t}$, we have

$$d_A(X^t_\lambda, Y) \leq d_A(X, Y) - t\,.$$

On the other end, by the triangle inequality we have $d_A(X^t_\lambda, Y) \geq d_A(X, Y) - t$. We thus have

$$d_A(X^t_\lambda, Y) = d_A(X, Y) - t\,.$$

This implies that μ is ratio-maximizing for X^t_λ and Y. If we choose N_X to be small enough, all other laminations in this neighborhood must intersect μ. We saw that $\mu(X^t_\lambda, Y) \subset N_X$ and this implies

$$\mu(X^t_\lambda, Y) = \mu\,.$$

■

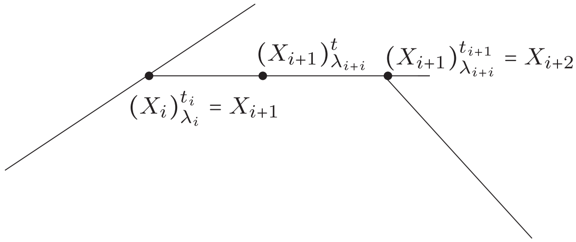


Fig. 24. Theorem 1.3.

Let $\bar{t} := \bar{t}_{X,Y,\lambda}$ be the supremum of the ϵ's as in Lemma 9.8. If $\bar{t} = d_A(X,Y)$, Lemma 9.8 gives a geodesic segment joining X and Y. Otherwise, we will need the following:

Lemma 9.9. If $\bar{t} < d_A(X,Y)$ then $\mu(X,Y) \subsetneq \mu(X^{\bar{t}}_\lambda, Y)$.

Proof. Let $\{t_n\}$ be a sequence of positive numbers such that $t_n \nearrow \bar{t}$. By Lemma 9.8 $\mu(X^{t_n}_\lambda, Y) = \mu(X,Y)$ for every n. Now, Lemma 9.7 says that $\mu(X^{t_n}_\lambda, Y) \subseteq \mu(X^{\bar{t}}_\lambda, Y)$.

By contradiction assume that $\mu(X,Y) = \mu(X^{\bar{t}}_\lambda, Y)$. Applying Lemma 9.8 on $X^{\bar{t}}_\lambda$ and Y, we find values bigger than $\bar{t}$ satisfying the same properties. ■

Theorem 1.3. The space $(\mathrm{Teich}(S), d_A)$ is a geodesic metric space. Every two points $X, Y \in \mathrm{Teich}(S)$ can be joined by a segment that is geodesic for both d_A and $d_{L\partial}$ and is a finite concatenation of generalized stretch segments.

Proof. We define inductively a sequence of hyperbolic structures $X_0, X_1, \ldots, X_k$ in the following way. We set $X_0 = X$. Now assume that X_i, $i \geq 0$ has been defined. Choose a maximal lamination λ_i that contains $\mu(X_i, Y)$. Consider the generalized stretch line $(X_i)^t_{\lambda_i}$, and compute $\bar{t}_i := \bar{t}_{X_i,Y,\lambda_i}$ as defined after Lemma 9.8. If $\bar{t}_i < d_A(X_i, Y)$, we set $X_{i+1} = (X_i)^{\bar{t}_i}_{\lambda_i}$, as in Fig. 24. If $\bar{t}_i \geq d_A(X_i, Y)$, this implies that Y lies on the generalized stretch line $(X_i)^t_{\lambda_i}$, in this case we set $k = i$ and we stop.

This defines the sequence of the X_i's. Notice that $\mu(X_i, Y) \subsetneq \mu(X_{i+1}, Y)$, so we have a strictly increasing chain of geodesic laminations. This implies $k \leq 2|\chi(S)|$. We have found a finite sequence of concatenated geodesic segments

$$t \to (X_i)^t_{\lambda_i}, \text{ for } 0 \leq t \leq \bar{t}_i$$

such that X lies in the 1st one, and Y in the last one. By Lemma 9.8, we have $d_A(X, X_i) + d_A(X_i, Y) = d_A(X, Y)$, hence this concatenation of segments is geodesic. ■

Corollary 1.5. Given $X, Y \in \mathrm{Teich}(S)$, there exists a continuous map $\phi \in \mathrm{Lip}_0(X, Y)$, with $\phi(\partial X) \subset \partial Y$ and with optimal Lipschitz constant such that $\log(\mathrm{Lip}(\phi)) = d_A(X, Y)$. In particular, we have

$$d_A(X, Y) = d_{L\partial}(X, Y).$$

Proof. Consider the geodesic segment joining X and Y in the proof of Theorem 1.3: it passes through the points $X = X_0, X_1, \ldots, X_k = Y$, where $X_{i+1} = (X_i)^{\bar{t}_i}_{\lambda_i}$. Since X_i and X_{i+1} are on the same stretch line, by Theorem 1.1 we have a map $\Phi_i^{\bar{t}_i} : X_i \to X_{i+1}$ with $\mathrm{Lip}(\Phi_i^{\bar{t}_i}) = e^{\bar{t}_i}$ and $\Phi_i^{\bar{t}_i}(\partial X_i) \subset \partial X_{i+1}$. Consider the composition:

$$\phi = \Phi_{k-1}^{\bar{t}_{k-1}} \circ \cdots \circ \Phi_0^{\bar{t}_0} : X \to Y\,,$$

which satisfies $\phi(\partial X) \subset \partial Y$. The Lipschitz constant of a composition is bounded by the product of the constants:

$$\mathrm{Lip}(\phi) \le \prod_i e^{\bar{t}_i} = e^{\sum_i \bar{t}_i} = e^{d_A(X,Y)}\,.$$

We know that $d_A(X, Y) \le d_{L\partial}(X, Y) \le \log(\mathrm{Lip}(\phi)) \le d_A(X, Y)$. Hence, $\log(\mathrm{Lip}(\phi)) = d_A(X, Y)$ and $d_A(X, Y) = d_{L\partial}(X, Y)$. ■

9.3 Geodesics in the Teichmüller space of the double

A *geodesic embedding* between two metric spaces $f : (\Omega, d) \to (\Omega', d')$ is an isometric embedding such that for every pair of points $P, Q \in f(\Omega)$ there exists a geodesic for d' that joins them and it is contained in $f(\Omega)$. The following is a consequence of Theorem 1.3 and Proposition 2.10.

Corollary 1.6. The map $(\mathrm{Teich}(S), d_A) \hookrightarrow (\mathrm{Teich}(S^d), d_{Th})$ is a geodesic embedding.

Corollary 1.7. Let $X \in \mathrm{Teich}(S)$ and let λ be a maximal lamination of X containing a measurable sublamination with at least one leaf orthogonal to the boundary of X. Then, the line $t \mapsto (X^t_\lambda)^d \in \mathrm{Teich}(S^d)$ is a geodesic for $(\mathrm{Teich}(S^d), d_{Th})$ that is not a stretch line in the sense of Thurston [20].

Proof. This line is a geodesic by Proposition 2.10. Notice that it stretches the length of the lamination λ^d by a factor e^t. If it were a stretch line in the sense of Thurston, it would be directed by a maximal lamination that contains λ^d. Since λ contains a leaf orthogonal to ∂X, every extension of λ^d to a maximal lamination in X^d is not symmetric, and the corresponding stretch line does not lie in the submanifold of symmetric hyperbolic structures (see Théret [19]). ■

In this way we find infinitely many examples of new geodesics for the Teichmüller spaces of surfaces without boundary that are not stretch lines in the sense of Thurston.

9.4 The Finsler metric

A *Finsler metric* on a smooth manifold M is a continuous function

$$F : TM \ni (x, v) \to F_x(v) \in \mathbb{R}_{\geq 0}$$

which is a (possibly asymmetric) norm on the tangent space T_xM at every point $x \in M$. In a Finsler manifold the length of a smooth curve $\gamma : [a, b] \to M$ is given by the formula

$$\ell(\gamma) := \int_a^b F_{\gamma(t)}(\dot{\gamma}(t))dt,$$

and the (possibly asymmetric) distance induced by a Finsler metric is defined as

$$d_F(x, y) = \inf_{\gamma} \ell(\gamma),$$

where the infimum is taken over all the smooth curves joining x and y.

Corollary 1.4. The arc distance d_A is induced by a Finsler metric on Teich(S).

Proof. Consider the map Teich(S) $\hookrightarrow$ Teich(S^d) as in Corollary 1.6. By [20], the space (Teich(S^d), d_{Th}) is a Finsler manifold. The space Teich(S) can be identified with a submanifold of Teich(S^d), and naturally inherits the Finsler metric by restriction. Now let's prove that the distance induced is the same as the distance d_A. Let X, Y be two points in Teich(S), we proved that $d_A(X, Y)$ is the same as the length of a geodesic segment joining them. By Corollary 1.6, the length of any geodesic segment in Teich(S)

is the same as in Teich(S^d). This in turn equals the length of the curve with respect to the Finsler norm, because the Finsler norm induces the distance d_{Th} on Teich(S^d). ■

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